# Chapter 22
# The Electric Field II: Continuous Charge Distributions

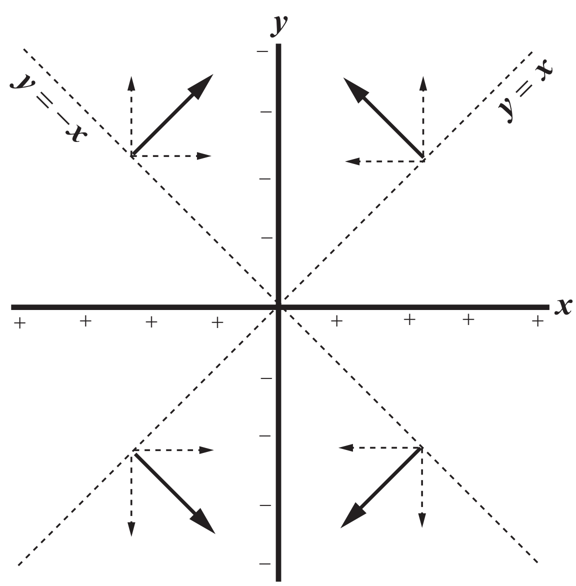
## Conceptual Problems

**1** • **[SSM]** Figure 22-37 shows an L-shaped object that has sides which are equal in length. Positive charge is distributed uniformly along the length of the object. What is the direction of the electric field along the dashed 45° line? Explain your answer.

**Determine the Concept** The resultant field is directed along the dashed line; pointing away from the intersection of the two sides of the L-shaped object. This can be seen by dividing each leg of the object into 10 (or more) equal segments and then drawing the electric field on the dashed line due to the charges on each pair of segments that are equidistant from the intersection of the legs.

**2** • Positive charge is distributed uniformly along the entire length of the $x$ axis, and negative charge is distributed uniformly along the entire length of the $y$ axis. The charge per unit length on the two axes is identical, except for the sign. Determine the direction of the electric field at points on the lines defined by $y = x$ and $y = -x$. Explain your answer.

**Determine the Concept** The electric fields along the lines defined by $y = x$ and $y = -x$ are the superposition of the electric fields due to the charge distributions along the axes. The direction of the electric field is the direction of the force acting on a test charge at the point(s) of interest. Typical points are shown at two points on each of the two lines.



**3** • True or false:

(*a*) The electric field due to a hollow uniformly charged thin spherical shell is zero at all points inside the shell.
(*b*) In electrostatic equilibrium, the electric field everywhere inside the material of a conductor must be zero.
(*c*) If the net charge on a conductor is zero, the charge density must be zero at every point on the surface of the conductor.

(*a*) True (assuming there are no charges inside the shell).

(*b*) True. The charges reside on the surface of conductor.

(*c*) False. Consider a spherical conducting shell. Such a surface will have equal charges on its inner and outer surfaces but, because their areas differ, so will their charge densities.

**4** • If the electric flux through a closed surface is zero, must the electric field be zero everywhere on that surface? If not, give a specific example. From the given information can the net charge inside the surface be determined? If so, what is it?

**Determine the Concept** No, this is not necessarily true. The only conclusion that we can draw is that there is equal positive and negative flux. For example, the net flux through a Gaussian surface completely enclosing a dipole is zero. If the electric flux is zero through the closed surface, we can conclude that the net charge inside the surface is zero.

**5** • True or false:

(*a*) Gauss's law holds only for symmetric charge distributions.
(*b*) The result that $E = 0$ everywhere inside the material of a conductor under electrostatic conditions can be derived from Gauss's law.

(*a*) False. Gauss's law states that the net flux through any surface is given by $\phi_{\text{net}} = \oint_S E_n dA = 4\pi k Q_{\text{inside}}$. While it is true that Gauss's law is easiest to apply to symmetric charge distributions, it holds for *any* surface.

(*b*) True. Because the charges on a conductor, under electrostatic conditions, reside on the surface of the conductor, the net flux inside the conductor is zero. Hence, by Gauss's law, the electric field inside the conductor must also be zero.

**6** •• A single point charge $q$ is located at the center of both an imaginary cube and an imaginary sphere. How does the electric flux through the surface of the cube compare to that through the surface of the sphere? Explain your answer.

**Determine the Concept** Because the net flux is proportional to the net charge enclosed, and this is the same for both surfaces, the electric flux through the surface of the cube is the same as the electric flux through the surface of the sphere.

**7** •• **[SSM]** An electric dipole is completely inside a closed imaginary surface and there are no other charges. True or False:

(*a*) The electric field is zero everywhere on the surface.
(*b*) The electric field is normal to the surface everywhere on the surface.
(c) The electric flux through the surface is zero.
(*d*) The electric flux through the surface could be positive or negative.
(*e*) The electric flux through a portion of the surface might not be zero.

(*a*) False. Near the positive end of the dipole, the electric field, in accordance with Coulomb's law, will be directed outward and will be nonzero. Near the negative end of the dipole, the electric field, in accordance with Coulomb's law, will be directed inward and will be nonzero.

(*b*) False. The electric field is perpendicular to the Gaussian surface only at the intersections of the surface with a line defined by the axis of the dipole.

(*c*) True. Because the net charge enclosed by the Gaussian surface is zero, the net flux, given by $\phi_{\text{net}} = \oint_S E_n dA = 4\pi k Q_{\text{inside}}$, through this surface must be zero.

(*d*) False. The flux through the closed surface is zero.

(*e*) True. All Gauss's law tells us is that, because the net charge inside the surface is zero, the *net* flux through the surface must be zero.

**8** •• Explain why the electric field strength increases linearly with $r$, rather than decreases inversely with $r^2$, between the center and the surface of a uniformly charged solid sphere.

**Determine the Concept** We can show that the charge inside a uniformly charged solid sphere of radius $r$ is proportional to $r^3$ and that the area of a sphere is proportional to $r^2$. Using Gauss's law, it follows that the electric field must be proportional to $r^3/r^2 = r$.

Use Gauss's law to express the electric field inside a spherical charge distribution of constant volume charge density:

$$E = \frac{4\pi k Q_{\text{inside}}}{A}$$

where $A = 4\pi r^2$.

Express $Q_{\text{inside}}$ as a function of $\rho$ and $r$:

$$Q_{\text{inside}} = \rho V = \tfrac{4}{3}\pi\rho r^3$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E = \frac{4\pi k \tfrac{4}{3}\pi\rho r^3}{4\pi r^2} = \frac{4k\pi\rho}{3} r$$

This result shows that the electric field *increases* linearly as you move out from the center of a spherical charge distribution.

**9** •• **[SSM]** Suppose that the total charge on the conducting spherical shell in Figure 22-38 is zero. The negative point charge at the center has a magnitude given by $Q$. What is the direction of the electric field in the following regions? (*a*) $r < R_1$ , (*b*) $R_2 > r > R_1$ , (*c*) and $r > R_2$ . Explain your answer.

**Determine the Concept** We can apply Gauss's law to determine the electric field for $r < R_1$, $R_2 > r > R_1$, and $r > R_2$. We also know that the direction of an electric field at any point is determined by the direction of the electric force acting on a positively charged object located at that point.

(*a*) From the application of Gauss's law we know that the electric field in this region is not zero. A positively charged object placed in the region for which $r < R_1$ will experience an attractive force from the charge $-Q$ located at the center of the shell. Hence the direction of the electric field is radially inward.

(*b*) Because the total charge on the conducting sphere is zero, the charge on its inner surface must be positive (the positive charges in the conducting sphere are drawn there by the negative charge at the center of the shell) and the charge on its outer surface must be negative. Hence the electric field in the region $R_2 > r > R_1$ is radially outward.

(*c*) Because the charge on the outer surface of the conducting shell is negative, the electric field in the region $r > R_2$ is radially inward.

**10** •• The conducting shell in Figure 22-38 is grounded, and the negative point charge at the center has a magnitude given by $Q$. Which of the following statements is correct?

(*a*) The charge on the inner surface of the shell is $+Q$ and the charge on the outer surface is $-Q$.
(*b*) The charge on the inner surface of the shell is $+Q$ and the charge on the outer surface is zero.
(*c*) The charge on both surfaces of the shell is $+Q$.
(*d*) The charge on both surfaces of the shell is zero.

**Determine the Concept** We can decide what will happen when the conducting shell is grounded by thinking about the distribution of charge on the shell before it is grounded and the effect on this distribution of grounding the shell.

The negative point charge at the center of the conducting shell induces a positive charge on the inner surface of the shell and a negative charge on the outer surface. Grounding the shell attracts positive charge from ground; resulting in the outer surface becoming electrically neutral. $\boxed{(b)}$ is correct.

**11** •• **T**he conducting shell in Figure 22-38 is grounded, and the negative point charge at the center has a magnitude given by $Q$. What is the direction of the electric field in the following regions? (*a*) $r < R_1$ , (*b*) $R_2 > r > R_1$ , (*c*) and $r > R_2$. Explain your answers.

**Determine the Concept** We can apply Gauss's law to determine the electric field for $r < R_1$, $R_2 > r > R_1$, and $r > R_2$. We also know that the direction of an electric field at any point is determined by the direction of the electric force acting on a positively charged object located at that point.

(*a*) From the application of Gauss's law we know that the electric field in this region is not zero. A positively charged object placed in the region for which $r < R_1$ will experience an attractive force from the charge $-Q$ located at the center of the shell. Hence the direction of the electric field is radially inward.

(*b*) Because the conducting shell is grounded, its inner surface is positively charged and its outer surface will have zero net charge. Hence the electric field in the region $R_2 > r > R_1$ is radially outward.

(*c*) Because the conducting shell is grounded, the net charge on the outer surface of the conducting shell is zero, and the electric field in the region $r > R_2$ is zero.

## Estimation and Approximation

**12** •• In the chapter, the expression for the electric field due to a uniformly charged disk (on its axis), was derived. At any location on the axis, the field

magnitude is $|E| = 2\pi k\sigma\left[1-\left(1+\frac{R^2}{z^2}\right)^{-1}\right]$. At large distances ($|z| >> R$), it was shown that this equation approaches $E \approx kQ/z^2$ . Very near the disk ($|z| << R$), the field strength is approximately that of an infinite plane of charge or $|E| \approx 2\pi k\sigma$ . Suppose you have a disk of radius 2.5 cm that has a uniform surface charge density of 3.6 $\mu$C/m$^2$. Use both the exact and appropriate expression from those given above to find the electric-field strength on the axis at distances of (*a*) 0.010 cm, (*b*) 0.040 cm, and (*c*) 5.0 m. Compare the two values in each case and comment on the how well the approximations work in their region of validity.

**Picture the Problem** For $z << R$, we can model the disk as an infinite plane. For $z >> R$, we can approximate the ring charge by a point charge.

(*a*) Evaluate the exact expression for $z$ = 0.010 cm:

$$|E|_{z=0.010\,\text{cm}} = 2\pi\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\frac{1}{\sqrt{1+\frac{(2.5\ \text{cm})^2}{(0.010\ \text{cm})^2}}}\right)$$

$$= 2.025\times10^5\ \text{N/C} = \boxed{2.0\times10^5\ \text{N/C}}$$

For $z << R$, the electric field strength near an infinite plane of charge is given by:

$$|E| \approx 2\pi k\sigma$$

Evaluate the approximate expression for $z$ = 0.010 cm:

$$|E|_{\text{approx}} = 2\pi\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right) = 2.033\times10^5\ \text{N/C}$$

$$= \boxed{2.0\times10^5\ \text{N/C}}$$

The approximate value agrees to within 0.40% with the exact value and is larger than the exact value.

(*b*) Evaluate the exact expression for $z$ = 0.040 cm:

$$\left|E\right|_{z=0.040\,\text{cm}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\frac{1}{\sqrt{1+\frac{(2.5\ \text{cm})^2}{(0.040\ \text{cm})^2}}}\right)$$

$$= 2.001\times10^{5}\ \text{N/C} = \boxed{2.0\times10^{5}\ \text{N/C}}$$

The approximate value agrees to within 1.2% with the exact value and is smaller than the exact value.

(*c*) Evaluate the exact expression for $z$ = 5.0 m:

$$\left|E\right|_{z=5.0\,\text{m}} = 2\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(3.6\,\mu\text{C/m}^2\right)\left(1-\frac{1}{\sqrt{1+\frac{(2.5\ \text{cm})^2}{(5.0\ \text{m})^2}}}\right)$$

$$= 2.541\ \text{N/C} = \boxed{2.5\ \text{N/C}}$$

Because $z >> R$, we can use Coulomb's law for the electric field due to a point charge to obtain:

$$\left|E(z)\right| = \frac{kQ}{z^2} = \frac{k\pi r^2\sigma}{z^2}$$

Evaluate $\left|E(5.0\ \text{m})\right|$:

$$\left|E(5.0\,\text{m})\right|_{\text{approx}} = \frac{\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.5\,\text{cm})^2\left(3.6\,\mu\text{C/m}^2\right)}{(5.0\,\text{m})^2} = 2.541\,\text{N/C}$$

$$= \boxed{2.5\,\text{N/C}}$$

The approximate value agrees, to four significant figures, with the exact value.

## Calculating $\vec{E}$ From Coulomb's Law

**13 •• [SSM]** A uniform line charge that has a linear charge density *l* equal to 3.5 nC/m is on the *x* axis between *x* = 0 and *x* = 5.0 m. (*a*) What is its total charge? Find the electric field on the *x* axis at (*b*) *x* = 6.0 m, (*c*) *x* = 9.0 m, and (*d*) *x* = 250 m.

(*e*) Estimate the electric field at $x = 250$ m, using the approximation that the charge is a point charge on the $x$ axis at $x = 2.5$ m, and compare your result with the result calculated in Part (*d*). (To do this you will need to assume that the values given in this problem statement are valid to more than two significant figures.) Is your approximate result greater or smaller than the exact result? Explain your answer.

**Picture the Problem** We can use the definition of $\lambda$ to find the total charge of the line of charge and the expression for the electric field on the axis of a finite line of charge to evaluate $E_x$ at the given locations along the $x$ axis. In Part (*d*) we can apply Coulomb's law for the electric field due to a point charge to approximate the electric field at $x = 250$ m.

(*a*) Use the definition of linear charge density to express $Q$ in terms of $\lambda$:

$$Q = \lambda L = (3.5\,\text{nC/m})(5.0\,\text{m}) = 17.5\,\text{nC} = \boxed{18\,\text{nC}}$$

Express the electric field on the axis of a finite line charge:

$$E_x(x_0) = \frac{kQ}{x_0(x_0 - L)}$$

(*b*) Substitute numerical values and evaluate $E_x$ at $x = 6.0$ m:

$$E_x(6.0\,\text{m}) = \frac{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(17.5\,\text{nC})}{(6.0\,\text{m})(6.0\,\text{m} - 5.0\,\text{m})} = \boxed{26\,\text{N/C}}$$

(*c*) Substitute numerical values and evaluate $E_x$ at $x = 9.0$ m:

$$E_x(9.0\,\text{m}) = \frac{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(17.5\,\text{nC})}{(9.0\,\text{m})(9.0\,\text{m} - 5.0\,\text{m})} = \boxed{4.4\,\text{N/C}}$$

(*d*) Substitute numerical values and evaluate $E_x$ at $x = 250$ m:

$$E_x(250\,\text{m}) = \frac{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(17.5\,\text{nC})}{(250\,\text{m})(250\,\text{m} - 5.0\,\text{m})} = 2.56800\,\text{mN/C} = \boxed{2.6\,\text{mN/C}}$$

(*e*) Use Coulomb's law for the electric field due to a point charge to obtain:

$$E_x(x) = \frac{kQ}{x^2}$$

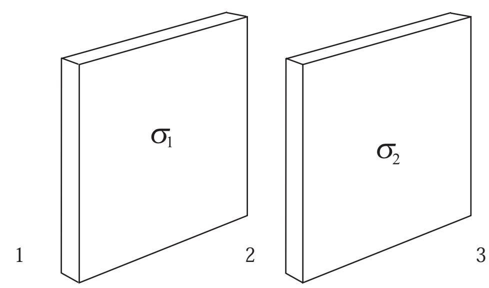
Substitute numerical values and evaluate $E_x(250\text{ m})$:

$$E_x(250\,\text{m}) = \frac{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(17.5\,\text{nC})}{(250\,\text{m}-2.5\,\text{m})^2} = 2.56774\,\text{mN/C} = \boxed{2.6\,\text{mN/C}}$$

This result is about 0.01% less than the exact value obtained in (*d*). This suggests that the line of charge is too long for its field at a distance of 250 m to be modeled exactly as that due to a point charge.

**14** • Two infinite non-conducting sheets of charge are parallel to each other, with sheet A in the $x = -2.0$ m plane and sheet B in the $x = +2.0$ m plane. Find the electric field in the region $x < -2.0$ m, in the region $x > +2.0$ m, and between the sheets for the following situations. (*a*) When each sheet has a uniform surface charge density equal to $+3.0\ \mu\text{C/m}^2$ and (*b*) when sheet A has a uniform surface charge density equal to $+3.0\ \mu\text{C/m}^2$ and sheet B has a uniform surface charge density equal to $-3.0\ \mu\text{C/m}^2$. (*c*) Sketch the electric field-line pattern for each case.

**Picture the Problem** Let the charge densities on the two plates be $\sigma_1$ and $\sigma_2$ and denote the three regions of interest as 1, 2, and 3. Choose a coordinate system in which the positive $x$ direction is to the right. We can apply the equation for $\vec{E}$ near an infinite plane of charge and the superposition of fields to find the field in each of the three regions.




(*a*) Use the equation for $\vec{E}$ near an infinite plane of charge to express the field in region 1 when $\sigma_1 = \sigma_2 = +3.0\ \mu\text{C/m}^2$:

$$\begin{aligned}\vec{E}_1 &= \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} \\ &= -2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i} \\ &= -4\pi k\sigma\,\hat{i}\end{aligned}$$

Substitute numerical values and evaluate $\vec{E}_1$:

$$\vec{E}_1 = -4\pi(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(3.0\,\mu\text{C/m}^2)\hat{i} = \boxed{-(3.4\times10^5\ \text{N/C})\hat{i}}$$

Proceed as above for region 2:

$$\vec{E}_2 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i} = 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i} = \boxed{0}$$

Proceed as above for region 3:

$$\vec{E}_3 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} + 2\pi k\sigma_2\hat{i} = 4\pi k\sigma\,\hat{i}$$
$$= 4\pi\left(8.988\times10^9\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left(3.0\ \mu\mathrm{C/m^2}\right)\hat{i}$$
$$= \boxed{\left(3.4\times10^5\ \mathrm{N/C}\right)\hat{i}}$$

(*b*) Use the equation for $\vec{E}$ near an infinite plane of charge to express and evaluate the field in region 1 when $\sigma_1 = +3.0\ \mu\mathrm{C/m^2}$ and $\sigma_2 = -3.0\ \mu\mathrm{C/m^2}$:

$$\vec{E}_1 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i} = 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i} = \boxed{0}$$

Proceed as above for region 2:

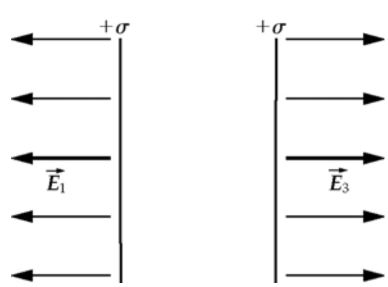
$$\vec{E}_2 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} + 2\pi k\sigma_2\hat{i} = 4\pi k\sigma\,\hat{i}$$
$$= 4\pi\left(8.988\times10^9\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left(3.0\ \mu\mathrm{C/m^2}\right)\hat{i}$$
$$= \boxed{\left(3.4\times10^5\ \mathrm{N/C}\right)\hat{i}}$$

Proceed as above for region 3:

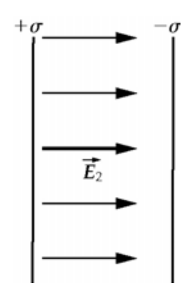
$$\vec{E}_3 = \vec{E}_{\sigma_1} + \vec{E}_{\sigma_2} = 2\pi k\sigma_1\hat{i} - 2\pi k\sigma_2\hat{i} = 2\pi k\sigma\,\hat{i} - 2\pi k\sigma\,\hat{i} = \boxed{0}$$

(*c*) The electric field lines for (*a*) and (*b*) are shown below:

(*a*)




(*b*)



**15** • A charge of 2.75 $\mu$C is uniformly distributed on a ring of radius 8.5 cm. Find the electric field strength on the axis at distances of (*a*) 1.2 cm, (*b*) 3.6 cm, and (*c*) 4.0 m from the center of the ring. (*d*) Find the field strength at 4.0 m using the approximation that the ring is a point charge at the origin, and compare your results for Parts (*c*) and (*d*). Is your approximate result a good one? Explain your answer.

**Picture the Problem** The magnitude of the electric field on the axis of a ring of charge is given by $E_x(z) = kQx\big/\left(z^2 + a^2\right)^{3/2}$ where $Q$ is the charge on the ring and $a$ is the radius of the ring. We can use this relationship to find the electric field on the axis of the ring at the given distances from the ring.

Express $\vec{\boldsymbol{E}}$ on the axis of a ring charge:

$$E_x(z) = \frac{kQx}{\left(z^2 + a^2\right)^{3/2}}$$

(*a*) Substitute numerical values and evaluate $E_x$ for $z$ = 1.2 cm:

$$\boldsymbol{E}_x(1.2\,\text{cm}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(1.2\,\text{cm})}{\left[(1.2\,\text{cm})^2 + (8.5\,\text{cm})^2\right]^{3/2}} = \boxed{4.7\times10^5\ \text{N/C}}$$

(*b*) Proceed as in (*a*) with $z$ = 3.6 cm:

$$\boldsymbol{E}_x(3.6\,\text{cm}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(3.6\,\text{cm})}{\left[(3.6\,\text{cm})^2 + (8.5\,\text{cm})^2\right]^{3/2}} = \boxed{1.1\times10^6\ \text{N/C}}$$

(*c*) Proceed as in (*a*) with $z$ = 4.0 m:

$$\boldsymbol{E}_x(4.0\,\text{m}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})(4.0\,\text{m})}{\left[(4.0\,\text{m})^2 + (8.5\,\text{cm})^2\right]^{3/2}} = \boxed{1.5\times10^3\ \text{N/C}}$$

(*d*) Using Coulomb's law for the electric field due to a point charge, express $E_z$:

$$E_z(z) = \frac{kQ}{z^2}$$

Substitute numerical values and evaluate $E_x$ at $z$ = 4.0 m:

$$E_z(4.0\,\text{m}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(2.75\,\mu\text{C})}{(4.0\,\text{m})^2} = \boxed{1.5\times10^3\ \text{N/C}}$$

While this result agrees exactly, to two significant figures, with the result obtained in Part (*c*), it should be slightly larger because the point charge is nearer $x = 4.0$ m than is the ring of charge.

**16** • A non-conducting disk of radius $R$ lies in the $z = 0$ plane with its center at the origin. The disk has a uniform surface charge density $\sigma$. Find the value of $z$ for which $E_z = \sigma/(4\in_0)$. Note that at this distance, the magnitude of the electric-field strength is half the electric-field strength at points on the $x$ axis that are very close to the disk.

**Picture the Problem** The electric field on the axis of a disk charge is given by $E_z = 2\pi kq\left(1-\frac{x}{\sqrt{z^2+R^2}}\right)$. We can equate this expression and $E_z = \frac{1}{2}\sigma/(2\in_0)$ and solve for $z$.

Express the electric field on the axis of a disk charge:

$$E_z = 2\pi kq\left(1-\frac{x}{\sqrt{z^2+R^2}}\right)$$

We're given that:

$$E_z = \tfrac{1}{2}\sigma/(2\in_0) = \frac{\sigma}{4\in_0}$$

Equate these expressions:

$$\frac{\sigma}{4\in_0} = 2\pi k\sigma\left(1-\frac{x}{\sqrt{z^2+R^2}}\right)$$

Substituting for $k$ yields:

$$\frac{\sigma}{4\in_0} = 2\pi\left(\frac{1}{4\pi\in_0}\right)\sigma\left(1-\frac{x}{\sqrt{z^2+R^2}}\right)$$

Solve for $z$ to obtain:

$$z = \boxed{\frac{R}{\sqrt{3}}}$$

**17** • **[SSM]** A ring that has radius $a$ lies in the $z = 0$ plane with its center at the origin. The ring is uniformly charged and has a total charge $Q$. Find $E_z$ on the $z$ axis at (*a*) $z = 0.2a$, (*b*) $z = 0.5a$, (*c*) $z = 0.7a$, (*d*) $z = a$, and (*e*) $z = 2a$. (*f*) Use your results to plot $E_z$ versus $z$ for both positive and negative values of $z$. (Assume that these distances are exact.)

**Picture the Problem** We can use $E_z = 2\pi kq\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$ to find the electric field at the given distances from the center of the charged ring.

(*a*) Evaluate $E_z(0.2a)$:

$$E_z(0.2a)=\frac{kQ(0.2a)}{\left[(0.2a)^2+a^2\right]^{3/2}}=\boxed{0.189\frac{kQ}{a^2}}$$

(*b*) Evaluate $E_z(0.5a)$:

$$E_z(0.5a)=\frac{kQ(0.5a)}{\left[(0.5a)^2+a^2\right]^{3/2}}=\boxed{0.358\frac{kQ}{a^2}}$$

(*c*) Evaluate $E_z(0.7a)$:

$$E_z(0.7a)=\frac{kQ(0.7a)}{\left[(0.7a)^2+a^2\right]^{3/2}}=\boxed{0.385\frac{kQ}{a^2}}$$

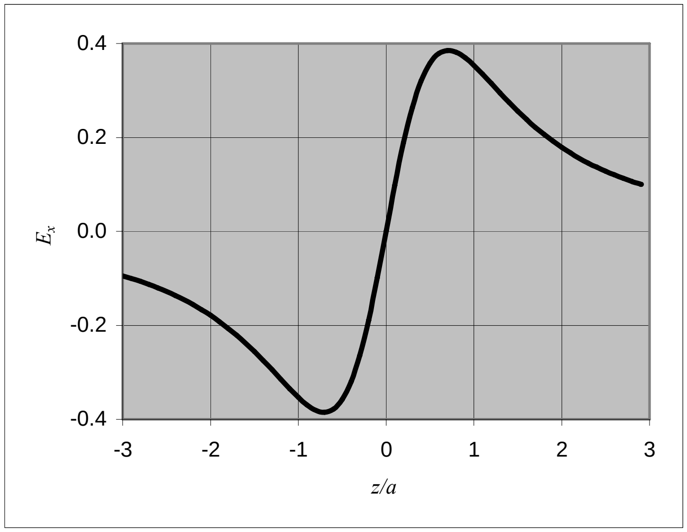
(*d*) Evaluate $E_z(a)$:

$$E_z(a)=\frac{kQa}{\left[a^2+a^2\right]^{3/2}}=\boxed{0.354\frac{kQ}{a^2}}$$

(*e*) Evaluate $E_z(2a)$:

$$E_z(2a)=\frac{2kQa}{\left[(2a)^2+a^2\right]^{3/2}}=\boxed{0.179\frac{kQ}{a^2}}$$

(*f*) The field along the $x$ axis is plotted below. The $z$ coordinates are in units of $z/a$ and $E$ is in units of $kQ/a^2$.




**18** • A non-conducting disk of radius $a$ lies in the $z = 0$ plane with its center at the origin. The disk is uniformly charged and has a total charge $Q$. Find $E_z$ on the $z$ axis at (*a*) $z = 0.2a$, (*b*) $z = 0.5a$, (*c*) $z = 0.7a$, (*d*) $z = a$, and (*e*) $z = 2a$. (*f*) Use your results to plot $E_z$ versus $z$ for both positive and negative values of $z$. (Assume that these distances are exact.)

**Picture the Problem** We can use $E_z = 2\pi kq\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)$, where $a$ is the radius of the disk, to find the electric field on the axis of a charged disk.

The electric field on the axis of a charged disk of radius $a$ is given by:

$$E_z = 2\pi kQ\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)$$
$$= \frac{Q}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)$$

(*a*) Evaluate $E_z(0.2a)$:

$$E_z(0.2a) = \frac{Q}{2\epsilon_0}\left(1 - \frac{0.2a}{\sqrt{(0.2a)^2 + a^2}}\right)$$
$$= \boxed{0.402\frac{Q}{\epsilon_0}}$$

(*b*) Evaluate $E_z(0.5a)$:

$$E_z(0.5a)=\frac{Q}{2\in_0}\left(1-\frac{0.5a}{\sqrt{(0.5a)^2+a^2}}\right)$$
$$=\boxed{0.276\frac{Q}{\in_0}}$$

(*c*) Evaluate $E_z(0.7a)$:

$$E_z(0.7a)=\frac{Q}{2\in_0}\left(1-\frac{0.7a}{\sqrt{(0.7a)^2+a^2}}\right)$$
$$=\boxed{0.213\frac{Q}{\in_0}}$$

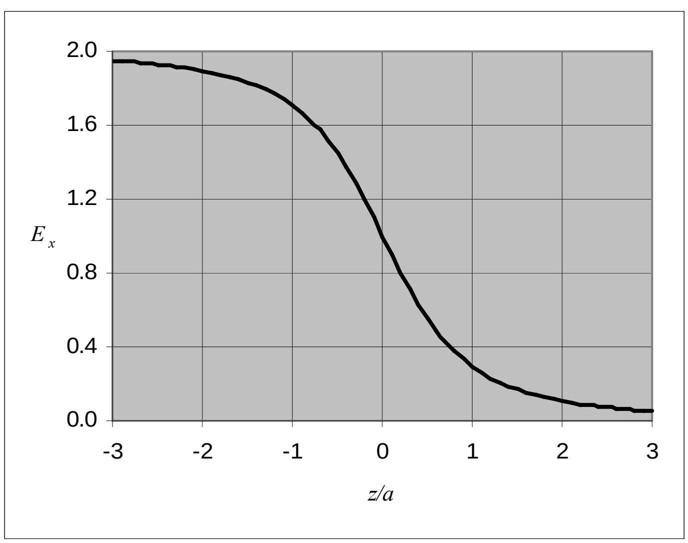
(*d*) Evaluate $E_z(a)$:

$$E_z(a)=\frac{Q}{2\in_0}\left(1-\frac{a}{\sqrt{a^2+a^2}}\right)$$
$$=\boxed{0.146\frac{Q}{\in_0}}$$

(*e*) Evaluate $E_z(2a)$:

$$E_z(2a)=\frac{Q}{2\in_0}\left(1-\frac{2a}{\sqrt{(2a)^2+a^2}}\right)$$
$$=\boxed{0.0528\frac{Q}{\in_0}}$$

The field along the *x* axis is plotted below. The *x* coordinates are in units of $z/a$ and *E* is in units of $Q/\in_0$.



**19** •• (*a*) Using a **spreadsheet** program or graphing calculator, make a graph of the electric field strength on the axis of a disk that has a radius $a = 30.0$ cm and a surface charge density $\sigma = 0.500$ nC/m$^2$. (*b*) Compare your results to the results based on the approximation $E = 2\pi k\sigma$ (the formula for the electric-field strength of a uniformly charged infinite sheet). At what distance does the solution based on approximation differ from the exact solution by 10.0 percent?

**Picture the Problem** The electric field on the *x* axis of a disk of radius *r* carrying a surface charge density $\sigma$ is given by $E_z = 2\pi k\sigma\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$. The electric field due to an infinite sheet of charge density $\sigma$ is independent of the distance from the plane and is given by $E_{\text{sheet}} = 2\pi k\sigma$.

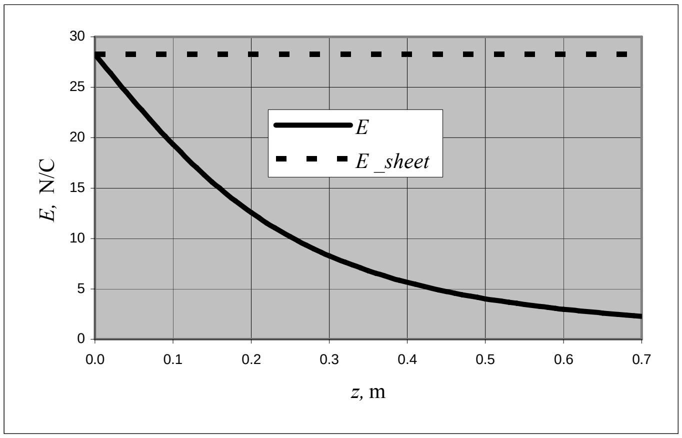
(*a*) A spreadsheet program to graph $E_x$ as a function of *x* is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Content/Formula | Algebraic Form |
|---|---|---|
| B3 | 9.00E+09 | $k$ |
| B4 | 5.00E−10 | $\sigma$ |
| B5 | 0.3 | $r$ |
| A8 | 0 | $x_0$ |
| A9 | 0.01 | $x_0 + 0.01$ |
| B8 | 2*PI()*$B$3*$B$4*(1−A8/ (A8^2+$B$5^2)^2)^0.5) | $2\pi k\sigma\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$ |
| C8 | 2*PI()*$B$3*$B$4 | $2\pi k\sigma$ |

| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | $k$= | 9.00E+09 | N·m$^2$/C$^2$ |
| 4 | $\sigma$= | 5.00E-10 | C/m$^2$ |
| 5 | $a$= | 0.300 | m |
| 6 | | | |
| 7 | $z$ | $E(z)$ | $E_{\text{sheet}}$ |
| 8 | 0.00 | 28.27 | 28.3 |
| 9 | 0.01 | 27.33 | 28.3 |
| 10 | 0.02 | 26.39 | 28.3 |
| 11 | 0.03 | 25.46 | 28.3 |
| 12 | 0.04 | 24.54 | 28.3 |
| 13 | 0.05 | 23.63 | 28.3 |
| 14 | 0.06 | 22.73 | 28.3 |
| 15 | 0.07 | 21.85 | 28.3 |
| | | | |
| 73 | 0.65 | 2.60 | 28.3 |

| | | | |
|---|---|---|---|
| 74 | 0.66 | 2.53 | 28.3 |
| 75 | 0.67 | 2.47 | 28.3 |
| 76 | 0.68 | 2.41 | 28.3 |
| 77 | 0.69 | 2.34 | 28.3 |
| 78 | 0.70 | 2.29 | 28.3 |

(*b*) The following graph shows $E$ as a function of $z$. The electric field from an infinite sheet with the same charge density is shown for comparison. The magnitudes differ by more than 10.0 percent for $x \geq 0.0300$ m.




**20** •• (*a*) Show that the electric-field strength $E$ on the axis of a ring charge of radius $a$ has maximum values at $z = \pm a/\sqrt{2}$. (*b*) Sketch the field strength $E$ versus $z$ for both positive and negative values of $z$. (*c*) Determine the maximum value of $E$ .

**Picture the Problem** The electric field on the axis of a ring charge as a function of distance $z$ along the axis from the center of the ring is given by $E_z = \dfrac{kQz}{\left(z^2 + a^2\right)^{3/2}}$. We can show that it has its maximum and minimum values at $z = +a/\sqrt{2}$ and $z = -a/\sqrt{2}$ by setting its first derivative equal to zero and solving the resulting equation for $z$. The graph of $E_z$ will confirm that the maximum and minimum occur at these coordinates.

(*a*) The variation of $E_z$ with $z$ on the axis of a ring charge is given by:

$$E_z = \frac{kQz}{\left(z^2 + a^2\right)^{3/2}}$$

Differentiate this expression with respect to $z$ to obtain:

$$\frac{dE_x}{dz} = kQ\frac{d}{dz}\left[\frac{x}{\left(z^2+a^2\right)^{3/2}}\right] = kQ\frac{\left(z^2+a^2\right)^{3/2} - z\frac{d}{dz}\left(z^2+a^2\right)^{3/2}}{\left(z^2+a^2\right)^3}$$

$$= kQ\frac{\left(z^2+a^2\right)^{3/2} - z\left(\tfrac{3}{2}\right)\left(z^2+a^2\right)^{1/2}(2z)}{\left(z^2+a^2\right)^3} = kQ\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^3}$$

Set this expression equal to zero for extrema and simplify:

$$\frac{\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2}}{\left(z^2+a^2\right)^3} = 0,$$

$$\left(z^2+a^2\right)^{3/2} - 3z^2\left(z^2+a^2\right)^{1/2} = 0,$$

and

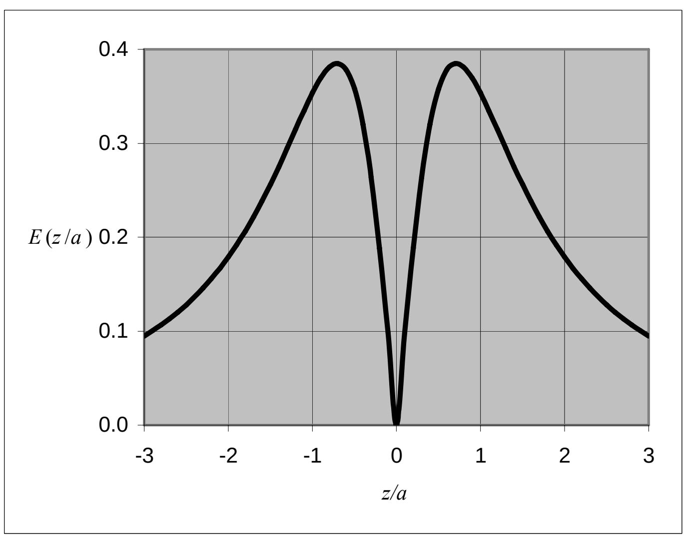
$$z^2 + a^2 - 3z^2 = 0$$

Solving for $z$ yields:

$$z = \boxed{\pm\frac{a}{\sqrt{2}}}$$

as our candidates for maxima or minima.

($b$) A plot of the magnitude of $E_z$, in units of $kQ/a^2$, versus $z/a$ follows. This graph shows that the extrema at $z = \pm a/\sqrt{2}$ are, in fact, maxima.



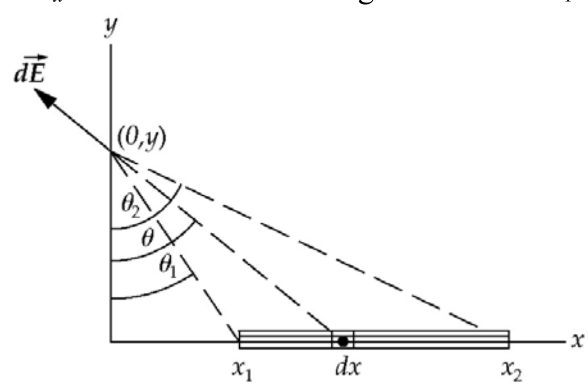
(*c*) Evaluate $E_z\left(\pm\frac{a}{\sqrt{2}}\right)$ and simplify to obtain the maximum value of the magnitude of $E_z$:

$$E_{z,\max}=E_z\left(\pm\frac{a}{\sqrt{2}}\right)=\left|\frac{kQ\left(\pm\frac{a}{\sqrt{2}}\right)}{\left(\left(\pm\frac{a}{\sqrt{2}}\right)^2+a^2\right)^{3/2}}\right|=\frac{kQ\frac{a}{\sqrt{2}}}{\left(\frac{1}{2}a^2+a^2\right)^{3/2}}=\boxed{\frac{2\sqrt{3}}{9}\frac{kQ}{a^2}}$$

**Remarks: Note that our result in Part (*c*) confirms the maxima obtained graphically in Part (*b*).**

**21** •• A line charge that has a uniform linear charge density $\lambda$ lies along the $x$ axis from $x = x_1$ to $x = x_2$ where $x_1 < x_2$. Show that the $x$ component of the electric field at a point on the $y$-axis is given by $E_x=\frac{k\lambda}{y}\left(\cos\theta_2-\cos\theta_1\right)$ where $\theta_1 = \tan^{-1}(x_1/y)$, $\theta_2 = \tan^{-1}(x_2/y)$ and $y \neq 0$.

**Picture the Problem** The line charge and point (0, $y$) are shown in the diagram. Also shown is a line element of length $dx$ and the field $d\vec{E}$ its charge produces at (0, $y$). We can find $dE_x$ from $d\vec{E}$ and then integrate from $x = x_1$ to $x = x_2$.




Express the $x$ component of $d\vec{E}$:

$$dE_x=-\frac{k\lambda}{x^2+y^2}\sin\theta dx$$
$$=-\frac{k\lambda}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}dx$$
$$=-\frac{k\lambda x}{\left(x^2+y^2\right)^{3/2}}dx$$

Integrate from $x = x_1$ to $x_2$ and simplify to obtain:

$$E_x = -k\lambda\int_{x_1}^{x_2}\frac{x}{\left(x^2+y^2\right)^{3/2}}dx$$

$$= -k\lambda\left[-\frac{1}{\sqrt{x^2+y^2}}\right]_{x_1}^{x_2}$$

$$= -k\lambda\left[-\frac{1}{\sqrt{x_2^2+y^2}}+\frac{1}{\sqrt{x_1^2+y^2}}\right]$$

$$= -\frac{k\lambda}{y}\left[-\frac{y}{\sqrt{x_2^2+y^2}}+\frac{y}{\sqrt{x_1^2+y^2}}\right]$$

From the diagram we see that:

$$\cos\theta_2 = \frac{y}{\sqrt{x_2^2+y^2}} \text{ or } \theta_2 = \tan^{-1}\left(\frac{x_2}{y}\right)$$

and

$$\cos\theta_1 = \frac{y}{\sqrt{x_1^2+y^2}} \text{ or } \theta_1 = \tan^{-1}\left(\frac{x_1}{y}\right)$$

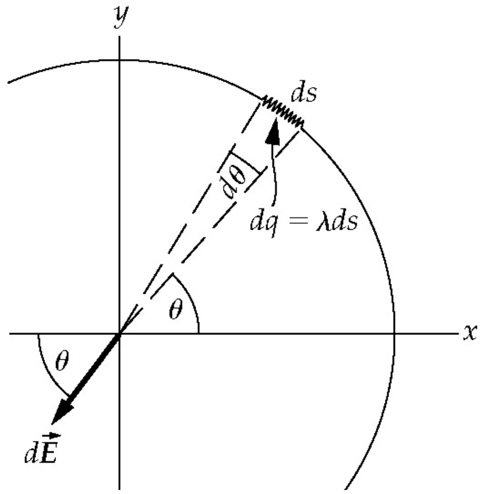
Substitute to obtain:

$$E_x = -\frac{k\lambda}{y}\left[-\cos\theta_2+\cos\theta_1\right]$$

$$= \boxed{\frac{k\lambda}{y}\left[\cos\theta_2-\cos\theta_1\right]}$$

**22** •• A ring of radius $a$ has a charge distribution on it that varies as $\lambda(\theta) = \lambda_0 \sin\theta$, as shown in Figure 22-39. ($a$) What is the direction of the electric field at the center of the ring? ($b$) What is the magnitude of the field at the center of the ring?

**Picture the Problem** The following diagram shows a segment of the ring of length $ds$ that has a charge $dq = \lambda ds$. We can express the electric field $d\vec{E}$ at the center of the ring due to the charge $dq$ and then integrate this expression from $\theta = 0$ to $2\pi$ to find the magnitude of the field in the center of the ring.




($a$) and ($b$) The field $d\vec{E}$ at the center of the ring due to the charge $dq$ is:

$$d\vec{E} = d\vec{E}_x + d\vec{E}_y = -dE\cos\theta\,\hat{i} - dE\sin\theta\,\hat{j} \qquad (1)$$

The magnitude $dE$ of the field at the center of the ring is:

$$dE = \frac{kdq}{r^2}$$

Because $dq = \lambda ds$:

$$dE = \frac{k\lambda ds}{r^2}$$

The linear charge density varies with $\theta$ according to $\lambda(\theta) = \lambda_0 \sin\theta$:

$$dE = \frac{k\lambda_0 \sin\theta\, ds}{r^2}$$

Substitute $rd\theta$ for $ds$:

$$dE = \frac{k\lambda_0 \sin\theta\, r d\theta}{r^2} = \frac{k\lambda_0 \sin\theta\, d\theta}{r}$$

Substitute for $dE$ in equation (1) to obtain:

$$d\vec{E} = -\frac{k\lambda_0 \sin\theta\cos\theta\, d\theta}{r}\hat{i} - \frac{k\lambda_0 \sin^2\theta\, d\theta}{r}\hat{j}$$

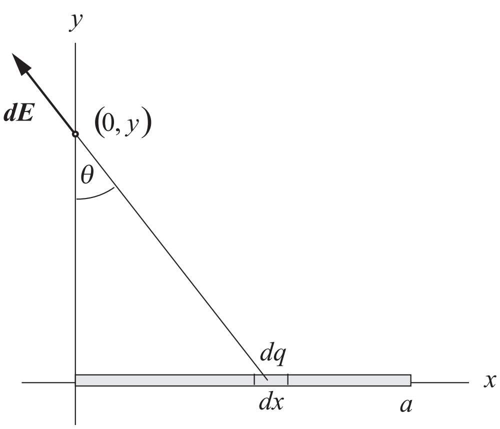
Integrate $d\vec{E}$ from $\theta = 0$ to $2\pi$ and simplify to obtain:

$$\vec{E} = -\frac{k\lambda_0}{2r}\int_0^{2\pi}\sin 2\theta\, d\theta\,\hat{i} - \frac{k\lambda_0}{r}\int_0^{2\pi}\sin^2\theta\, d\theta\,\hat{j} = 0 - \frac{\pi k\lambda_0}{r}\hat{j} = \boxed{-\frac{\pi k\lambda_0}{r}\hat{j}}$$

($b$) The field at the origin is in the negative $y$ direction and its magnitude is $\frac{\pi k\lambda_0}{r}$.

**23** •• A line of charge that has uniform linear charge density $\lambda$ lies on the $x$ axis from $x = 0$ to $x = a$. Show that the $y$ component of the electric field at a point on the $y$ axis is given by $E_y = \dfrac{k\lambda}{y}\dfrac{a}{\sqrt{y^2+a^2}}$, $y \neq 0$.

**Picture the Problem** The line of charge and the point whose coordinates are (0, $y$) are shown in the diagram. Also shown is a segment of the line of length $dx$ and charge $dq$. The field due to this charge at (0, $y$) is $d\vec{E}$. We can find $dE_y$ from $d\vec{E}$ and then integrate from $x$ = 0 to $x$ = $a$ to find the $y$ component of the electric field at a point on the $y$ axis.




($a$) Express the magnitude of the field $d\vec{E}$ due to charge $dq$ of the element of length $dx$:

$$dE = \frac{kdq}{r^2}$$

where $r^2 = x^2 + y^2$

Because $dq = \lambda dx$:

$$dE = \frac{k\lambda dx}{x^2 + y^2}$$

Express the $y$ component of $dE$:

$$dE_y = \frac{k\lambda}{x^2 + y^2}\cos\theta\, dx$$

Refer to the diagram to express $\cos\theta$ in terms of $x$ and $y$:

$$\cos\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Substitute for $\cos\theta$ in the expression for $dE_y$ to obtain:

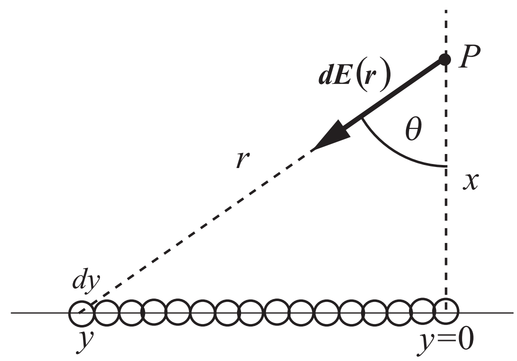
$$dE_y = \frac{k\lambda y}{\left(x^2 + y^2\right)^{3/2}}dx$$

Integrate from $x = 0$ to $x = a$ and simplify to obtain:

$$E_y = k\lambda y\int_0^a \frac{1}{\left(x^2 + y^2\right)^{3/2}}dx = k\lambda y\left[\frac{x}{y^2\sqrt{x^2 + y^2}}\right]_0^a = \boxed{\frac{k\lambda}{y}\frac{a}{\sqrt{a^2 + y^2}}}$$

**24** ••• Calculate the electric field a distance $z$ from a uniformly charged infinite flat non-conducting sheet by modeling the sheet as a continuum of infinite straight lines of charge.

**Picture the Problem** The field due to a line of charge is given by $E(r)=\frac{1}{2\pi\in_0}\frac{\lambda}{r}$ where $r$ is the perpendicular distance to the line. The diagram shows a point $P$, at which we will calculate the electric field due a continuum of infinite straight non-conducting lines of charge, and a few of the lines of charge. $P$ is a distance $L$ from the plane and the origin of the coordinate system is directly below $P$. Note that the horizontal components of the field at $P$, by symmetry, add up to zero. Hence we need only find the sum of all the $z$ components of the field.




Because the horizontal components of the electric field add up to zero, the resultant field is given by:

$$E=E_{\perp}=\int_{-\pi/2}^{\pi/2}dE(r)\cos\theta \qquad (1)$$

Express the field due to an infinite line of charge:

$dE(r)=\frac{1}{2\pi\in_0}\frac{d\lambda}{r}$, where $r$ is the perpendicular distance to the line of charge.

The surface charge density $\sigma$ of the plane and the linear charge density of the charged rings $\lambda$ are related:

$$d\lambda=\sigma\,dy$$

Substitute for $d\lambda$ to obtain:

$$dE(r)=\frac{1}{2\pi\in_0}\frac{\sigma\,dy}{r}$$

Substituting for $dE(r)$ in equation (1) yields:

$$E=\frac{1}{2\pi\in_0}\int_{-\pi/2}^{\pi/2}\frac{\sigma\,dy}{r}\cos\theta$$

Referring to the diagram, note that:

$$y=x\tan\theta\Rightarrow dy=x\sec^2\theta\,d\theta$$

Substitute for $dy$ in the expression for $E$ to obtain:

$$E = \frac{\sigma}{2\pi \in_0} \int_{-\pi/2}^{\pi/2} \frac{x\sec^2\theta\cos\theta\, d\theta}{r}$$

Because $\boldsymbol{x} = \boldsymbol{r}\cos\theta$:

$$E_\perp = \frac{\sigma}{2\pi \in_0} \int_{-\pi/2}^{\pi/2} \sec^2\theta\cos^2\theta\, d\theta$$

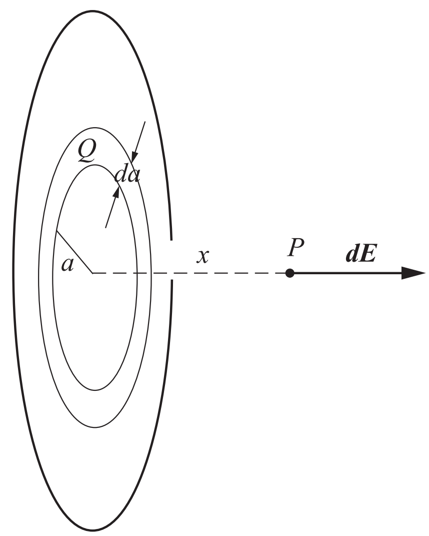
$$= \frac{\sigma}{2\pi \in_0} \int_{-\pi/2}^{\pi/2} d\theta$$

Integrating this expression yields:

$$E = \frac{\sigma}{2\pi \in_0} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\sigma}{2\pi \in_0}\pi = \boxed{\frac{\sigma}{2 \in_0}}$$

**25 •• [SSM]** Calculate the electric field a distance $z$ from a uniformly charged infinite flat non-conducting sheet by modeling the sheet as a continuum of infinite circular rings of charge.

**Picture the Problem** The field at a point on the axis of a uniformly charged ring lies along the axis and is given by Equation 22-8. The diagram shows one ring of the continuum of circular rings of charge. The radius of the ring is $a$ and the distance from its center to the field point $P$ is $x$. The ring has a uniformly distributed charge $Q$. The resultant electric field at $P$ is the sum of the fields due to the continuum of circular rings. Note that, by symmetry, the horizontal components of the electric field cancel.




Express the field of a single uniformly charged ring with charge $Q$ and radius $a$ on the axis of the ring at a distance $x$ away from the plane of the ring:

$$\vec{\boldsymbol{E}} = \boldsymbol{E}_x\hat{\boldsymbol{i}}\text{, where } \boldsymbol{E}_x = \frac{\boldsymbol{kQx}}{\left(\boldsymbol{x}^2 + \boldsymbol{a}^2\right)^{3/2}}$$

Substitute $dq$ for $Q$ and $dE_x$ for $E_x$ to obtain:

$$dE_x = \frac{kxdq}{\left(x^2 + a^2\right)^{3/2}}$$

The resultant electric field at $P$ is the sum of the fields due to all the circular rings. Integrate both sides to calculate the resultant field for the entire plane. The field point remains fixed, so $x$ is constant:

$$E = \int \frac{kx\,dq}{\left(x^2 + a^2\right)^{3/2}} = kx\int \frac{dq}{\left(x^2 + a^2\right)^{3/2}}$$

To evaluate this integral we change integration variables from $q$ to $a$. The charge $dq = \sigma\,dA$ where $dA = 2\pi\,a\,da$ is the area of a ring of radius $a$ and width $da$:

$$dq = 2\pi\sigma\,a\,da$$

so

$$E = kx\int_0^\infty \frac{2\pi\,\sigma\,a\,da}{\left(x^2 + a^2\right)^{3/2}} = 2\pi\,\sigma\,kx\int_0^\infty \frac{a\,da}{\left(x^2 + a^2\right)^{3/2}}$$

To integrate this expression, let $u = \sqrt{x^2 + a^2}$. Then:

$$du = \tfrac{1}{2}\frac{1}{\sqrt{x^2 + a^2}}\left(2a\,da\right) = \frac{a}{u}da$$

or

$$a\,da = u\,du$$

Noting that when $a = 0$, $u = x$, substitute and simplify to obtain:

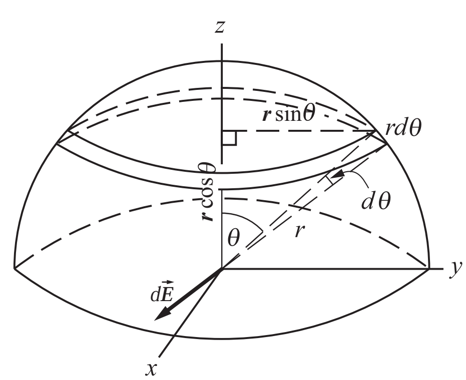
$$E = 2\pi\sigma\,kx\int_x^\infty \frac{u}{u^3}du = 2\pi\sigma\,kx\int_x^\infty u^{-2}du$$

Evaluating the integral yields:

$$E = 2\pi\sigma\,kx\left(-\frac{1}{u}\right)\Bigg|_x^\infty = 2\pi k\sigma = \boxed{\frac{\sigma}{2\epsilon_0}}$$

**26** ••• A thin hemispherical shell of radius $R$ has a uniform surface charge $\sigma$. Find the electric field at the center of the base of the hemispherical shell.

**Picture the Problem** Consider the ring with its axis along the $z$ direction shown in the diagram. Its radius is $z = r\cos\theta$ and its width is $rd\theta$. We can use the equation for the field on the axis of a ring charge and then integrate to express the field at the center of the hemispherical shell.



Express the field on the axis of the ring charge:

$$dE = \frac{kzdq}{\left(r^2\sin^2\theta + r^2\cos^2\theta\right)^{3/2}} = \frac{kzdq}{r^3}$$

where $z = r\cos\theta$

Express the charge $dq$ on the ring:

$$dq = \sigma dA = \sigma\left(2\pi r\sin\theta\right)rd\theta = 2\pi\sigma r^2\sin\theta d\theta$$

Substitute to obtain:

$$dE = \frac{k\left(r\cos\theta\right)2\pi\sigma r^2\sin\theta d\theta}{r^3} = 2\pi k\sigma\sin\theta\cos\theta d\theta$$

Integrating $dE$ from $\theta = 0$ to $\pi/2$ yields:

$$E = 2\pi k\sigma\int_0^{\pi/2}\sin\theta\cos\theta d\theta = 2\pi k\sigma\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \boxed{\pi k\sigma}$$

## Gauss's Law

**27** • A square that has 10-cm-long edges is centered on the $x$ axis in a region where there exists a uniform electric field given by $\vec{\boldsymbol{E}} = \left(2.00\ \text{kN/C}\right)\hat{\boldsymbol{i}}$ . ($a$) What is the electric flux of this electric field through the surface of a square if the normal to the surface is in the $+x$ direction? ($b$) What is the electric flux through the same square surface if the normal to the surface makes a 60º angle with the $y$ axis and an angle of 90° with the $z$ axis?

**Picture the Problem** The definition of electric flux is $\phi = \oint_S \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}dA$ . We can apply this definition to find the electric flux through the square in its two orientations.

($a$) Apply the definition of $\phi$ to find the flux of the field when the square is parallel to the $yz$ plane:

$$\phi = \oint_S\left(2.00\,\text{kN/C}\right)\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}}dA = \left(2.00\,\text{kN/C}\right)\oint_S dA = \left(2.00\,\text{kN/C}\right)\left(0.100\,\text{m}\right)^2 = \boxed{20.0\,\text{N}\cdot\text{m}^2/\text{C}}$$

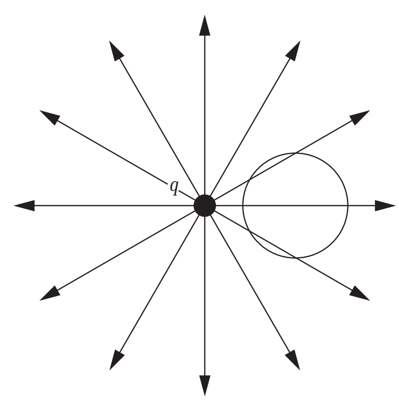
(*b*) Proceed as in (*a*) with $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}} = \cos 30°$:

$$\begin{aligned}\phi &= \oint_S (2.00\,\text{kN/C})\cos 30°\, dA \\ &= (2.00\,\text{kN/C})\cos 30° \oint_S dA \\ &= (2.00\,\text{kN/C})(0.100\,\text{m})^2 \cos 30° \\ &= \boxed{17\,\text{N}\cdot\text{m}^2/\text{C}}\end{aligned}$$

**28** • A single point charge ($q =$ +2.00 $\mu$C) is fixed at the origin. An imaginary spherical surface of radius 3.00 m is centered on the $x$ axis at $x = 5.00$ m. (*a*) Sketch electric-field lines for this charge (in two dimensions) assuming twelve equally-spaced field lines in the $xy$ plane leave the charge location, with one of the lines in the $+x$ direction. Do any lines enter the spherical surface? If so, how many? (*b*) Do any lines leave the spherical surface? If so, how many? (*c*) Counting the lines that enter as negative and the ones that leave as positive, what is the net number of field lines that penetrate the spherical surface? (*d*) What is the net electric flux through this spherical surface?

**Determine the Concept** We must show the twelve electric field lines originating at $q$ and, in the absence of other charges, radially symmetric with respect to the location of $q$. While we're drawing twelve lines in this problem, the number of lines that we draw is always, by agreement, in proportion to the magnitude of $q$.

(*a*) The sketch of the field lines and of the spherical surface is shown in the diagram to the right.




Given the number of field lines drawn from $q$, 3 lines enter the spherical surface. Had we chosen to draw 24 field lines, 6 would have entered the spherical surface.

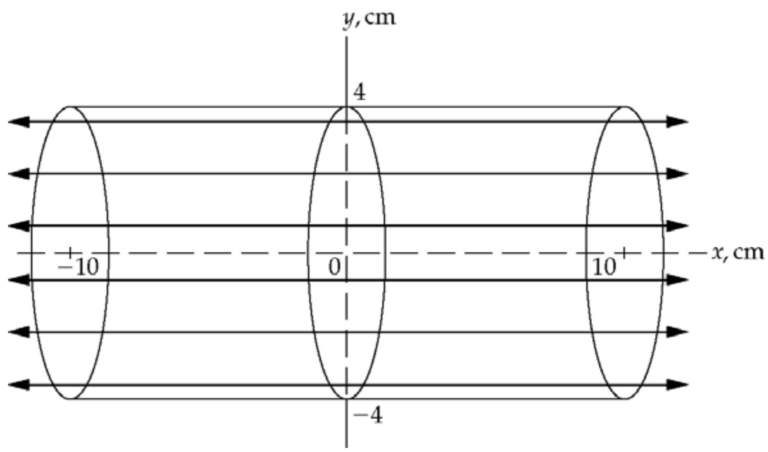
(*b*) Three lines leave the spherical surface.

(*c*) Because the three lines that enter the spherical surface also leave the spherical surface, the net number of field lines that pass through the surface is zero.

(*d*) Because as many field lines leave the spherical surface as enter it, the net flux is zero.

**29** • **[SSM]** An electric field is given by $\vec{E} = \text{sign}(x)\cdot(300\text{ N/C})\hat{i}$, where sign($x$) equals –1 if $x < 0$, 0 if $x = 0$, and +1 if $x > 0$. A cylinder of length 20 cm and radius 4.0 cm has its center at the origin and its axis along the $x$ axis such that one end is at $x = +10$ cm and the other is at $x = -10$ cm. ($a$) What is the electric flux through each end? ($b$) What is the electric flux through the curved surface of the cylinder? ($c$) What is the electric flux through the entire closed surface? ($d$) What is the net charge inside the cylinder?

**Picture the Problem** The field at both circular faces of the cylinder is parallel to the outward vector normal to the surface, so the flux is just $EA$. There is no flux through the curved surface because the normal to that surface is perpendicular to $\vec{E}$. The net flux through the closed surface is related to the net charge inside by Gauss's law.




($a$) Use Gauss's law to calculate the flux through the right circular surface:

$$\phi_{\text{right}} = \vec{E}_{\text{right}}\cdot\hat{n}_{\text{right}}A = (300\text{ N/C})\hat{i}\cdot\hat{i}(\pi)(0.040\text{ m})^2 = \boxed{1.5\text{ N}\cdot\text{m}^2/\text{C}}$$

Apply Gauss's law to the left circular surface:

$$\phi_{\text{left}} = \vec{E}_{\text{left}}\cdot\hat{n}_{\text{left}}A = (-300\text{ N/C})\hat{i}\cdot(-\hat{i})(\pi)(0.040\text{ m})^2 = \boxed{1.5\text{ N}\cdot\text{m}^2/\text{C}}$$

($b$) Because the field lines are parallel to the curved surface of the cylinder:

$$\phi_{\text{curved}} = \boxed{0}$$

(*c*) Express and evaluate the net flux through the entire cylindrical surface:

$$\phi_{\text{net}} = \phi_{\text{right}} + \phi_{\text{left}} + \phi_{\text{curved}}$$
$$= 1.5\,\text{N}\cdot\text{m}^2/\text{C} + 1.5\,\text{N}\cdot\text{m}^2/\text{C} + 0$$
$$= \boxed{3.0\,\text{N}\cdot\text{m}^2/\text{C}}$$

(*d*) Apply Gauss's law to obtain:

$$\phi_{\text{net}} = 4\pi k Q_{\text{inside}} \Rightarrow Q_{\text{inside}} = \frac{\phi_{\text{net}}}{4\pi k}$$

Substitute numerical values and evaluate $Q_{\text{inside}}$:

$$Q_{\text{inside}} = \frac{3.0\,\text{N}\cdot\text{m}^2/\text{C}}{4\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)}$$
$$= \boxed{2.7\times10^{-11}\ \text{C}}$$

**30** • Careful measurement of the electric field at the surface of a black box indicates that the net outward electric flux through the surface of the box is 6.0 kN·m$^2$/C. (*a*) What is the net charge inside the box? (*b*) If the net outward electric flux through the surface of the box were zero, could you conclude that there were no charges inside the box? Explain your answer.

**Picture the Problem** We can use Gauss's law in terms of $\epsilon_0$ to find the net charge inside the box.

(*a*) Apply Gauss's law in terms of $\epsilon_0$ to find the net charge inside the box:

$$\phi_{\text{net}} = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow Q_{\text{inside}} = \epsilon_0\,\phi_{\text{net}}$$

Substitute numerical values and evaluate $Q_{\text{inside}}$:

$$Q_{\text{inside}} = \left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(6.0\,\frac{\text{kN}\cdot\text{m}^2}{\text{C}}\right) = \boxed{5.3\times10^{-8}\ \text{C}}$$

(*b*) You can only conclude that the net charge is zero. There may be an equal number of positive and negative charges present inside the box.

**31** • A point charge ($q = +2.00\ \mu\text{C}$) is at the center of an imaginary sphere that has a radius equal to 0.500 m. (*a*) Find the surface area of the sphere. (*b*) Find the magnitude of the electric field at all points on the surface of the sphere. (*c*) What is the flux of the electric field through the surface of the sphere? (*d*) Would your answer to Part (*c*) change if the point charge were moved so that it was inside the sphere but not at its center? (*e*) What is the flux of the electric field through the surface of an imaginary cube that has 1.00-m-long edges and encloses the sphere?

**Picture the Problem** We can apply Gauss's law to find the flux of the electric field through the surface of the sphere.

(*a*) Use the formula for the surface area of a sphere to obtain:

$$A = 4\pi r^2 = 4\pi(0.500\,\text{m})^2 = 3.142\,\text{m}^2 = \boxed{3.14\,\text{m}^2}$$

(*b*) Apply Coulomb's law to find $E$:

$$E = \frac{1}{4\pi \in_0}\frac{q}{r^2} = \frac{1}{4\pi\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\frac{2.00\,\mu\text{C}}{(0.500\,\text{m})^2} = 7.190\times10^4\,\text{N/C}$$
$$= \boxed{7.19\times10^4\,\text{N/C}}$$

(*c*) Apply Gauss's law to obtain:

$$\phi = \oint_S \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}dA = \oint_S EdA$$
$$= \left(7.190\times10^4\,\text{N/C}\right)\left(3.142\,\text{m}^2\right)$$
$$= \boxed{2.26\times10^5\,\text{N}\cdot\text{m}^2/\text{C}}$$

(*d*) No. The flux through the surface is independent of where the charge is located inside the sphere.

(*e*) Because the cube encloses the sphere, the flux through the surface of the sphere will also be the flux through the cube:

$$\phi_{\text{cube}} = \boxed{2.26\times10^5\,\text{N}\cdot\text{m}^2/\text{C}}$$

**32** • What is the electric flux through one side of a cube that has a single point charge of –3.00 $\mu$C placed at its center? *HINT: You do not need to integrate any equations to get the answer.*

**Picture the Problem** The flux through the cube is given by $\phi_{\text{net}} = Q_{\text{inside}}/\in_0$, where $Q_{\text{inside}}$ is the charge at the center of the cube. The flux through one side of the cube is one-sixth of the total flux through the cube.

The flux through one side of the cube is one-sixth of the total flux through the cube:

$$\phi_{\text{1 face}} = \tfrac{1}{6}\phi_{\text{tot}} = \frac{Q}{6\in_0}$$

Substitute numerical values and evaluate $\phi_{1\text{ face}}$:

$$\phi_{1\text{ face}} = \frac{-3.00\ \mu\text{C}}{6\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)} = \boxed{-5.65\times10^{4}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}}}$$

**33 • [SSM]** A single point charge is placed at the center of an imaginary cube that has 20-cm-long edges. The electric flux out of one of the cube's sides is –1.50 kN·m$^2$/C. How much charge is at the center?

**Picture the Problem** The net flux through the cube is given by $\phi_{\text{net}} = Q_{\text{inside}}/\epsilon_0$, where $Q_{\text{inside}}$ is the charge at the center of the cube.

The flux through one side of the cube is one-sixth of the total flux through the cube:

$$\phi_{1\text{ faces}} = \tfrac{1}{6}\phi_{\text{net}} = \frac{Q_{\text{inside}}}{6\,\epsilon_0}$$

Solving for $Q_{\text{inside}}$ yields:

$$Q_{\text{inside}} = 6\,\epsilon_0\,\phi_{2\text{ faces}}$$

Substitute numerical values and evaluate $Q_{\text{inside}}$:

$$Q_{\text{inside}} = 6\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(-1.50\frac{\text{kN}\cdot\text{m}^2}{\text{C}^2}\right) = \boxed{-79.7\ \text{nC}}$$

**34 ••** Because the formulas for Newton's law of gravity and for Coulomb's law have the same inverse-square dependence on distance, a formula analogous to the formula for Gauss's law can be found for gravity. The gravitational field $\vec{\boldsymbol{g}}$ at a location is the force per unit mass on a test mass $m_0$ placed at that location. Then, for a point mass $m$ at the origin, the gravitational field $\vec{\boldsymbol{g}}$ at some position ($\vec{\boldsymbol{r}}$) is $\vec{\boldsymbol{g}} = \left(Gm/r^2\right)\hat{\boldsymbol{r}}$. Compute the flux of the gravitational field through a spherical surface of radius $R$ centered at the origin, and verify that the gravitational analog of Gauss's law is $\phi_{\text{net}} = -4\pi G m_{\text{inside}}$.

**Picture the Problem** We'll define the flux of the gravitational field in a manner that is analogous to the definition of the flux of the electric field and then substitute for the gravitational field and evaluate the integral over the closed spherical surface.

Define the gravitational flux as:

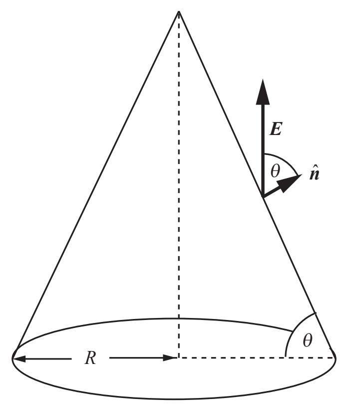
$$\phi_{\text{g}} = \oint_S \vec{\boldsymbol{g}}\cdot\hat{\boldsymbol{n}}\,dA$$

Substitute for $\vec{g}$ and evaluate the integral to obtain:

$$\phi_{\text{net}} = \oint_S \left(-\frac{Gm_{\text{inside}}}{r^2}\hat{\boldsymbol{r}}\right)\cdot\hat{\boldsymbol{n}}dA$$
$$= -\frac{Gm_{\text{inside}}}{r^2}\oint_S dA$$
$$= \left(-\frac{Gm_{\text{inside}}}{r^2}\right)\left(4\pi r^2\right)$$
$$= \boxed{-4\pi Gm_{\text{inside}}}$$

**35** •• An imaginary right circular cone (Figure 22-40) that has a base angle $\theta$ and a base radius $R$ is in a charge free region that has a uniform electric field $\vec{E}$ (field lines are vertical and parallel to the cone's axis). What is the ratio of the number of field lines per unit area penetrating the base to the number of field lines per unit area penetrating the conical surface of the cone? Use Gauss's law in your answer. (The field lines in the figure are only a representative sample.)

**Picture the Problem** Because the cone encloses no charge, we know, from Gauss's law, that the net flux of the electric field through the cone's surface is zero. Thus, the number of field lines penetrating the curved surface of the cone must equal the number of field lines penetrating the base and the entering flux must equal the exiting flux.




The flux penetrating the base of the cone is given by:

$$\boldsymbol{\phi}_{\text{entering}} = \boldsymbol{E}\boldsymbol{A}_{\text{base}}$$

The flux penetrating the curved surface of the cone is given by:

$$\boldsymbol{\phi}_{\text{exiting}} = \oint_S \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}\boldsymbol{dA} = \oint_S \boldsymbol{E}\cos\boldsymbol{\theta}\,\boldsymbol{dA}$$

Equating the fluxes and simplifying yields:

$$\boldsymbol{A}_{\text{base}} = \cos\boldsymbol{\theta}\oint_S \boldsymbol{dA} = (\cos\boldsymbol{\theta})\boldsymbol{A}_{\substack{\text{curved}\\ \text{surface}}}$$

The ratio of the density of field lines is:

$$\frac{\boldsymbol{A}_{\text{base}}}{\boldsymbol{A}_{\substack{\text{curved}\\ \text{surface}}}} = \boxed{\cos\boldsymbol{\theta}}$$

**36** •• In the atmosphere and at an altitude of 250 m, you measure the electric field to be 150 N/C directed downward and you measure the electric field to be 170 N/C directed downward at an altitude of 400 m. Calculate the volume

charge density of the atmosphere in the region between altitudes of 250 m and 400 m, assuming it to be uniform. (You may neglect the curvature of Earth. Why?)

**Picture the Problem** We'll model this portion of Earth's atmosphere as though it is a cylinder with cross-sectional area $A$ and height $h$. Because the electric flux increases with altitude, we can conclude that there is charge inside the cylindrical region and use Gauss's law to find that charge and hence the charge density of the atmosphere in this region.

The definition of volume charge density is:

$$\rho = \frac{Q}{V}$$

Express the charge inside a cylinder of base area $A$ and height $h$ for a charge density $\rho$:

$$Q = \rho A h$$

Taking upward to be the positive direction, apply Gauss's law to the charge in the cylinder:

$$Q = -\left(E_h A - E_0 A\right)\epsilon_0 = \left(E_0 A - E_h A\right)\epsilon_0$$

where we've taken our zero at 250 m above the surface of a flat Earth.

Substitute to obtain:

$$\rho = \frac{\left(E_0 A - E_h A\right)\epsilon_0}{Ah} = \frac{\left(E_0 - E_h\right)\epsilon_0}{h}$$

Substitute numerical values and evaluate $\rho$:

$$\rho = \frac{(150\,\text{N/C} - 170\,\text{N/C})\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}{400\,\text{m} - 250\,\text{m}} = \boxed{-1.2\times10^{-12}\,\text{C/m}^3}$$

where we've been able to neglect the curvature of Earth because the maximum height of 400 m is approximately 0.006% of the radius of Earth.

## Gauss's Law Applications in Spherical Symmetry Situations

**37** • A thin non-conducting spherical shell of radius $R_1$ has a total charge $q_1$ that is uniformly distributed on its surface. A second, larger thin non-conducting spherical shell of radius $R_2$ that is coaxial with the first has a charge $q_2$ that is uniformly distributed on its surface. (*a*) Use Gauss's law to obtain expressions for the electric field in each of the three regions: $r < R_1$, $R_1 < r < R_2$, and $r > R_2$. (*b*) What should the ratio of the charges $q_1/q_2$ and the relative signs for $q_1$ and $q_2$ be for the electric field to be zero throughout the region $r > R_2$? (*c*) Sketch the electric field lines for the situation in Part (*b*) when $q_1$ is positive.

**Picture the Problem** To find $E_n$ in these three regions we can choose Gaussian surfaces of appropriate radii and apply Gauss's law. On each of these surfaces, $E_r$ is constant and Gauss's law relates $E_r$ to the total charge inside the surface.

(*a*) Use Gauss's law to find the electric field in the region $r < R_1$:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

and

$$\vec{E}_{r<R_1} = \frac{Q_{\text{inside}}}{\epsilon_0 A}\hat{r}$$ where $\hat{r}$ is a unit radial vector.

Because $Q_{\text{inside}} = 0$:

$$\vec{E}_{r<R_1} = \boxed{0}$$

Apply Gauss's law in the region $R_1 < r < R_2$:

$$\vec{E}_{R_1<r<R_1} = \frac{q_1}{\epsilon_0\left(4\pi r^2\right)}\hat{r} = \boxed{\frac{kq_1}{r^2}\hat{r}}$$

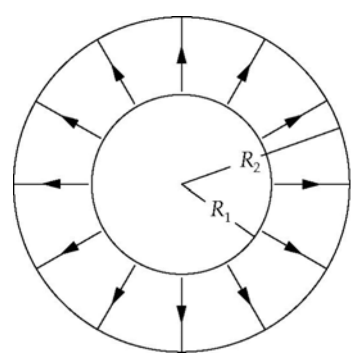
Using Gauss's law, find the electric field in the region $r > R_2$:

$$\vec{E}_{r>R_2} = \frac{q_1 + q_2}{\epsilon_0\left(4\pi r^2\right)}\hat{r} = \boxed{\frac{k\left(q_1 + q_2\right)}{r^2}\hat{r}}$$

(*b*) Set $E_{r>R_2} = 0$ to obtain:

$$q_1 + q_2 = 0 \Rightarrow \frac{q_1}{q_2} = \boxed{-1}$$

(*c*) The electric field lines for the situation in (*b*) with $q_1$ positive is shown to the right.




**38** • A spherical shell of radius 6.00 cm carries a uniform surface charge density of A non-conducting thin spherical shell of radius 6.00 cm has a uniform surface charge density of 9.00 nC/m$^2$. (*a*) What is the total charge on the shell? Find the electric field at the following distances from the sphere's center: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

**Picture the Problem** We can use the definition of surface charge density and the formula for the area of a sphere to find the total charge on the shell. Because the charge is distributed uniformly over a spherical shell, we can choose a spherical

Gaussian surface and apply Gauss's law to find the electric field as a function of the distance from the center of the spherical shell.

(*a*) Using the definition of surface charge density, relate the charge on the sphere to its area:

$$Q = \sigma A = 4\pi\sigma r^2$$

Substitute numerical values and evaluate $Q$:

$$Q = 4\pi\left(9.00\,\text{nC/m}^2\right)\left(0.0600\,\text{m}\right)^2 = 0.4072\,\text{nC} = \boxed{0.407\,\text{nC}}$$

Apply Gauss's law to a spherical surface of radius $r$ that is concentric the spherical shell to obtain:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_n$ yields:

$$E_n = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

(*b*) The charge inside a sphere whose radius is 2.00 cm is zero and hence:

$$E_n\left(2.00\,\text{cm}\right) = \boxed{0}$$

(*c*) The charge inside a sphere whose radius is 5.90 cm is zero and hence:

$$E_n\left(5.90\,\text{cm}\right) = \boxed{0}$$

(*d*) The charge inside a sphere whose radius is 6.10 cm is 0.4072 nC and hence:

$$E_n\left(6.10\,\text{cm}\right) = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0610\,\text{m}\right)^2} = \boxed{983\,\text{N/C}}$$

(*e*) The charge inside a sphere whose radius is 10.0 cm is 0.4072 nC and hence:

$$E_n\left(10\,\text{cm}\right) = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.100\,\text{m}\right)^2} = \boxed{366\,\text{N/C}}$$

**39 •• [SSM]** A non-conducting sphere of radius 6.00 cm has a uniform volume charge density of 450 nC/m$^3$. (*a*) What is the total charge on the sphere? Find the electric field at the following distances from the sphere's center: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

**Picture the Problem** We can use the definition of volume charge density and the formula for the volume of a sphere to find the total charge of the sphere. Because the charge is distributed uniformly throughout the sphere, we can choose a spherical Gaussian surface and apply Gauss's law to find the electric field as a function of the distance from the center of the sphere.

(*a*) Using the definition of volume charge density, relate the charge on the sphere to its volume:

$$Q = \rho V = \tfrac{4}{3}\pi\rho r^3$$

Substitute numerical values and evaluate $Q$:

$$Q = \tfrac{4}{3}\pi\left(450\,\text{nC/m}^3\right)\left(0.0600\,\text{m}\right)^3 = 0.4072\,\text{nC} = \boxed{0.407\,\text{nC}}$$

Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the spherical shell to obtain:

$$\oint_S E_n\,dA = \frac{1}{\epsilon_0}Q_{\text{inside}} \Rightarrow 4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_n$ yields:

$$E_n = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

Because the charge distribution is uniform, we can find the charge inside the Gaussian surface by using the definition of volume charge density to establish the proportion:

$$\frac{Q}{V} = \frac{Q_{\text{inside}}}{V'}$$

where $V'$ is the volume of the Gaussian surface.

Solve for $Q_{\text{inside}}$ to obtain:

$$Q_{\text{inside}} = Q\frac{V'}{V} = Q\frac{r^3}{R^3}$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E_n(r < R) = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ}{R^3}r$$

(*b*) Evaluate $E_n$ at $r = 2.00$ cm:

$$\boldsymbol{E}_n(2.00\,\text{cm}) = \frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(0.4072\,\text{nC}\right)}{\left(0.0600\,\text{m}\right)^3}\left(0.0200\,\text{m}\right) = \boxed{339\,\text{N/C}}$$

($c$) Evaluate $E_n$ at $r$ = 5.90 cm:

$$E_n(5.90\,\text{cm}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(0.4072\,\text{nC})}{(0.0600\,\text{m})^3}(0.0590\,\text{m}) = \boxed{1.00\,\text{kN/C}}$$

Apply Gauss's law to the Gaussian surface with $r > R$:

$$4\pi r^2 E_n = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow E_n = \frac{kQ_{\text{inside}}}{r^2} = \frac{kQ}{r^2}$$

($d$) Evaluate $E_n$ at $r$ = 6.10 cm:

$$E_n(6.10\,\text{cm}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(0.4072\,\text{nC})}{(0.0610\,\text{m})^2} = \boxed{983\,\text{N/C}}$$

($e$) Evaluate $E_n$ at $r$ = 10.0 cm:

$$E_n(10.0\,\text{cm}) = \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(0.4072\,\text{nC})}{(0.100\,\text{m})^2} = \boxed{366\,\text{N/C}}$$

**40** •• Consider the solid conducting sphere and the concentric conducting spherical shell in Figure 22-41. The spherical shell has a charge $-7Q$. The solid sphere has a charge $+2Q$. ($a$) How much charge is on the outer surface and how much charge is on the inner surface of the spherical shell? ($b$) Suppose a metal wire is now connected between the solid sphere and the shell. After electrostatic equilibrium is re-established, how much charge is on the solid sphere and on each surface of the spherical shell? Does the electric field at the surface of the solid sphere change when the wire is connected? If so, in what way? ($c$) Suppose we return to the conditions in Part ($a$), with $+2Q$ on the solid sphere and $-7Q$ on the spherical shell. We next connect the solid sphere to ground with a metal wire, and then disconnect it. Then how much total charge is on the solid sphere and on each surface of the spherical shell?

**Determine the Concept** The charges on a conducting sphere, in response to the repulsive Coulomb forces each experiences, will separate until electrostatic equilibrium conditions exit. The use of a wire to connect the two spheres or to ground the outer sphere will cause additional redistribution of charge.

($a$) Because the outer sphere is conducting, the field in the thin shell must vanish. Therefore, $-2Q$, uniformly distributed, resides on the inner surface, and $-5Q$, uniformly distributed, resides on the outer surface.

(*b*) Now there is no charge on the inner surface and −5*Q* on the outer surface of the spherical shell. The electric field just outside the surface of the inner sphere changes from a finite value to zero.

(*c*) In this case, the −5*Q* is drained off, leaving no charge on the outer surface and −2*Q* on the inner surface. The total charge on the outer sphere is then −2*Q*.

**41** •• A non-conducting solid sphere of radius 10.0 cm has a uniform volume charge density. The magnitude of the electric field at 20.0 cm from the sphere's center is $1.88 \times 10^3$ N/C. (*a*) What is the sphere's volume charge density? (*b*) Find the magnitude of the electric field at a distance of 5.00 cm from the sphere's center.

**Picture the Problem** (*a*) We can use the definition of volume charge density, in conjunction with Equation 22-18a, to find the sphere's volume charge density. (*b*) We can use Equation 22-18b, in conjunction with our result from Part (*a*), to find the electric field at a distance of 5.00 cm from the solid sphere's center.

(*a*) The solid sphere's volume charge density is the ratio of its charge to its volume:

$$\rho = \frac{Q_{\text{inside}}}{V} = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi R^3} \qquad (1)$$

For $r \geq R$, Equation 22-18*a* gives the electric field at a distance *r* from the center of the sphere:

$$E_{\text{r}} = \frac{1}{4\pi \epsilon_0}\frac{Q_{\text{inside}}}{r^2} \qquad (2)$$

Solving for $Q_{\text{inside}}$ yields:

$$Q_{\text{inside}} = 4\pi \epsilon_0 E_{\text{r}} r^2$$

Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

$$\rho = \frac{4\pi \epsilon_0 E_{\text{r}} r^2}{\frac{4}{3}\pi R^3} = \frac{3 \epsilon_0 E_{\text{r}} r^2}{R^3}$$

Substitute numerical values and evaluate $\rho$:

$$\rho = \frac{3\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)\left(1.88\times10^3\ \text{N/C}\right)\left(20.0\ \text{cm}\right)^2}{\left(10.0\ \text{cm}\right)^3} = 1.997\,\mu\text{C/m}^3$$

$$= \boxed{2.00\,\mu\text{C/m}^3}$$

(*b*) For $r \leq R$, the electric field at a distance *r* from the center of the sphere is given by:

$$E_{\text{r}} = \frac{1}{4\pi \epsilon_0}\frac{Q_{\text{inside}}}{R^3} r \qquad (3)$$

Express $Q_{\text{inside}}$ for $r \le R$:

$$Q_{\text{inside}} = \rho V_{\substack{\text{sphere whose}\\ \text{radius is } r}} = \tfrac{4}{3}\pi r^3 \rho$$

Substituting for $Q_{\text{inside}}$ in equation (3) and simplifying yields:

$$E_{\text{r}} = \frac{1}{4\pi \in_0} \frac{\tfrac{4}{3}\pi r^3 \rho}{R^3} r = \frac{\rho r^4}{3 \in_0 R^3}$$

Substitute numerical values and evaluate $E_r(5.00 \text{ cm})$:

$$E_{\text{r}}(5.00\text{ cm}) = \frac{\left(1.997\,\mu\text{C/m}^3\right)\left(5.00\text{ cm}\right)^4}{3\left(8.854\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2\right)\left(10.0\text{ cm}\right)^3} = \boxed{470\text{ N/C}}$$

**42** •• A non-conducting solid sphere of radius $R$ has a volume charge density that is proportional to the distance from the center. That is, $\rho = Ar$ for $r \le R$, where $A$ is a constant. (*a*) Find the total charge on the sphere. (*b*) Find the expressions for the electric field inside the sphere ($r < R$) and outside the sphere ($r > R$). (*c*) Sketch the magnitude of the electric field as a function of the distance $r$ from the sphere's center.

**Picture the Problem** We can find the total charge on the sphere by expressing the charge $dq$ in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find $E_{\text{r}}$ inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find $E_{\text{r}}$ outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, $E_{\text{r}}$ is constant. Gauss's law then relates $E_{\text{r}}$ to the total charge inside the surface.

(*a*) Express the charge $dq$ in a shell of thickness $dr$ and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 (Ar) dr = 4\pi A r^3 dr$$

Integrate this expression from $r = 0$ to $R$ to find the total charge on the sphere:

$$Q = 4\pi A \int_0^R r^3 dr = \left[\pi A r^4\right]_0^R = \boxed{\pi A R^4}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_{\text{r}} dA = \frac{1}{\in_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_{\text{r}} = \frac{Q_{\text{inside}}}{\in_0}$$

Solving for $E_r$ yields:

$$E_r(r > R) = \frac{Q_{\text{inside}}}{4\pi \in_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

$$= \frac{kA\pi R^4}{r^2} = \boxed{\frac{AR^4}{4 \in_0 r^2}}$$

Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

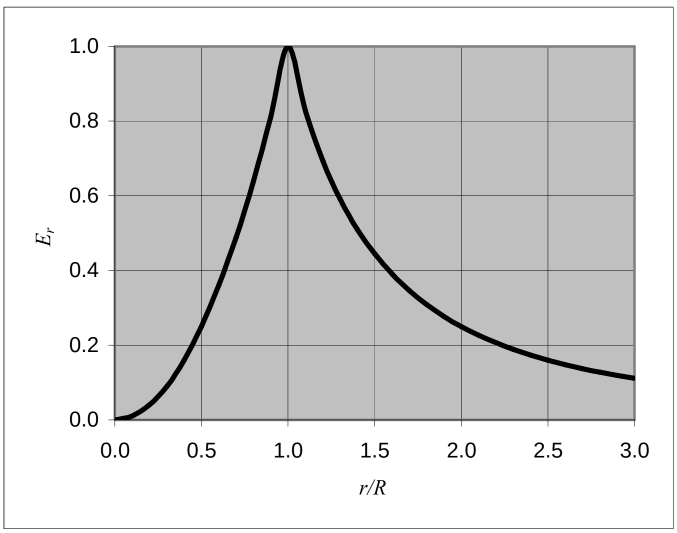
$$\oint_S E_r dA = \frac{1}{\in_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\in_0}$$

Solve for $E_r$ to obtain:

$$E_r(r < R) = \frac{Q_{\text{inside}}}{4\pi r^2 \in_0} = \frac{\pi A r^4}{4\pi r^2 \in_0}$$

$$= \boxed{\frac{Ar^2}{4 \in_0}}$$

(*c*) The following graph of $E_r$ versus $r/R$, with $E_r$ in units of $A/(4\in_0)$, was plotted using a spreadsheet program.




**Remarks: Note that the results for (*a*) and (*b*) agree at *r* = *R*.**

**43 •• [SSM] A** sphere of radius $R$ has volume charge density $\rho = B/r$ for $r < R$, where $B$ is a constant and $\rho = 0$ for $r > R$. (*a*) Find the total charge on the sphere. (*b*) Find the expressions for the electric field inside and outside the charge distribution (c) Sketch the magnitude of the electric field as a function of the distance $r$ from the sphere's center.

**Picture the Problem** We can find the total charge on the sphere by expressing the charge $dq$ in a spherical shell and integrating this expression between $r = 0$ and

$r = R$. By symmetry, the electric fields must be radial. To find $E_r$ inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find $E_r$ outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, $E_r$ is constant. Gauss's law then relates $E_r$ to the total charge inside the surface.

(*a*) Express the charge $dq$ in a shell of thickness $dr$ and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 \frac{B}{r} dr$$
$$= 4\pi B r dr$$

Integrate this expression from $r = 0$ to $R$ to find the total charge on the sphere:

$$Q == 4\pi B \int_0^R r dr = \left[2\pi B r^2\right]_0^R$$
$$= \boxed{2\pi B R^2}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \text{ or } 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r(r > R) = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$
$$= \frac{k 2\pi B R^2}{r^2} = \boxed{\frac{BR^2}{2\epsilon_0 r^2}}$$

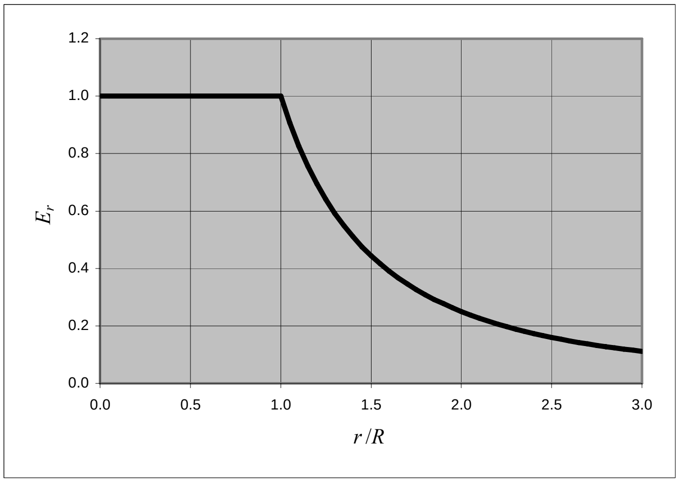
Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r(r < R) = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} = \frac{2\pi B r^2}{4\pi r^2 \epsilon_0}$$
$$= \boxed{\frac{B}{2\epsilon_0}}$$

($c$) The following graph of $E_r$ versus $r/R$, with $E_r$ in units of $B/(2\in_0)$, was plotted using a spreadsheet program.




**Remarks: Note that our results for (*a*) and (*b*) agree at *r* = *R*.**

**44** •• **A** sphere of radius $R$ has volume charge density $\rho = C/r^2$ for $r < R$, where $C$ is a constant and $\rho = 0$ for $r > R$. ($a$) Find the total charge on the sphere. ($b$) Find the expressions for the electric field inside and outside the charge distribution ($c$) Sketch the magnitude of the electric field as a function of the distance $r$ from the sphere's center.

**Picture the Problem** We can find the total charge on the sphere by expressing the charge $dq$ in a spherical shell and integrating this expression between $r = 0$ and $r = R$. By symmetry, the electric fields must be radial. To find $E_\mathrm{r}$ inside the charged sphere we choose a spherical Gaussian surface of radius $r < R$. To find $E_\mathrm{r}$ outside the charged sphere we choose a spherical Gaussian surface of radius $r > R$. On each of these surfaces, $E_\mathrm{r}$ is constant. Gauss's law then relates $E_\mathrm{r}$ to the total charge inside the surface.

($a$) Express the charge $dq$ in a shell of thickness $dr$ and volume $4\pi r^2\,dr$:

$$dq = 4\pi r^2 \rho dr = 4\pi r^2 \frac{C}{r^2} dr = 4\pi C dr$$

Integrate this expression from $r = 0$ to $R$ to find the total charge on the sphere:

$$Q = 4\pi C\int_0^R dr = \left[4\pi C r\right]_0^R = \boxed{4\pi CR}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r > R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r(r > R) = \frac{Q_{\text{inside}}}{4\pi \epsilon_0} \frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$
$$= \frac{k4\pi CR}{r^2} = \boxed{\frac{CR}{\epsilon_0 r^2}}$$

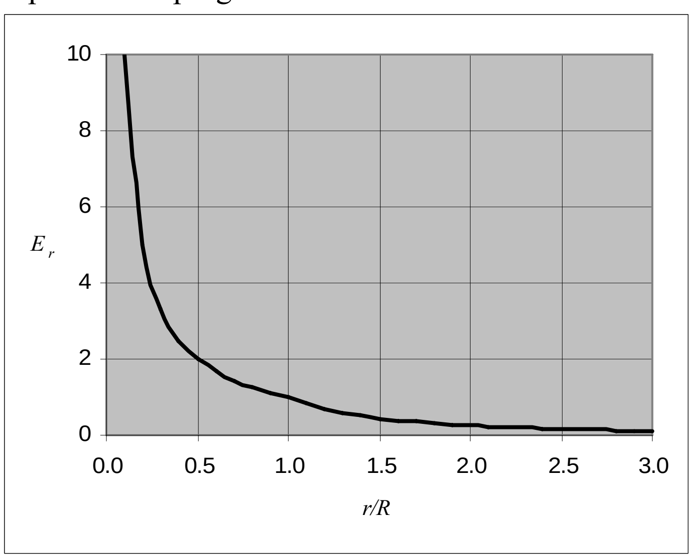
Apply Gauss's law to a spherical surface of radius $r < R$ that is concentric with the nonconducting sphere to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \text{ or } 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r(r < R) = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} = \frac{4\pi Cr}{4\pi r^2 \epsilon_0}$$
$$= \boxed{\frac{C}{\epsilon_0 r}}$$

(*c*) The following graph of $E_r$ versus $r/R$, with $E_r$ in units of $C/(\epsilon_0 R)$, was plotted using a spreadsheet program.




**45** ••• A non-conducting spherical shell of inner radius $R_1$ and outer radius $R_2$ has a uniform volume charge density $\rho$. (*a*) Find the total charge on the shell. (*b*) Find expressions for the electric field everywhere.

**Picture the Problem** By symmetry, the electric fields resulting from this charge distribution must be radial. To find $E_r$ for $r < R_1$ we choose a spherical Gaussian surface of radius $r < R_1$. To find $E_r$ for $R_1 < r < R_2$ we choose a spherical Gaussian surface of radius $R_1 < r < R_2$. To find $E_r$ for $r > R_2$ we choose a spherical Gaussian surface of radius $r > R_2$. On each of these surfaces, $E_r$ is constant. Gauss's law then relates $E_r$ to the total charge inside the surface.

(*a*) The charge in an infinitesimal spherical shell of radius *r* and thickness *dr* is:

$$dQ = \rho dV = 4\pi\rho r^2 dr$$

Integrate *dQ* from $r = R_1$ to *r* to find the total charge in the spherical shell in the interval $R_1 < r < R_2$:

$$Q_{\text{inside}} = 4\pi\rho\int_{R_1}^{r} r^2 dr = \left[\frac{4\pi C r^3}{3}\right]_{R_1}^{r}$$

$$= \boxed{\frac{4\pi\rho}{3}\left(r^3 - R_1^3\right)}$$

(*b*) Apply Gauss's law to a spherical surface of radius *r* that is concentric with the nonconducting spherical shell to obtain:

$$\oint_S E_r dA = \frac{1}{\epsilon_0}Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r(r) = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2}$$

Evaluate $E_r(r < R_1)$:

$$E_r(r < R_1) = \frac{Q_{\text{inside}}}{4\pi\epsilon_0}\frac{1}{r^2} = \frac{kQ_{\text{inside}}}{r^2} = \boxed{0}$$

because $\rho(r < R_1) = 0$ and, therefore, $Q_{\text{inside}} = 0$.

Evaluate $E_r(R_1 < r < R_2)$:

$$E_r(R_1 < r < R_2) = \frac{kQ_{\text{inside}}}{r^2}$$

$$= \frac{4\pi k\rho}{3r^2}\left(R_2^3 - R_1^3\right)$$

$$= \boxed{\frac{\rho}{3\epsilon_0 r^2}\left(R_2^3 - R_1^3\right)}$$

For $r > R_2$:

$$Q_{\text{inside}} = \frac{4\pi\rho}{3}\left(R_2^3 - R_1^3\right)$$

and

$$E_{\text{r}}\left(r > R_2\right) = \frac{4\pi k\rho}{3r^2}\left(R_2^3 - R_1^3\right)$$

$$= \boxed{\frac{\rho}{3\in_0 r^2}\left(R_2^3 - R_1^3\right)}$$

**Remarks: Note that *E* is continuous at *r* = *R*₂.**

## Gauss's Law Applications in Cylindrical Symmetry Situations

**46** • For your senior project you are in charge of designing a Geiger tube for detecting radiation in the nuclear physics laboratory. This instrument will consist of a long metal cylindrical tube that has a long straight metal wire running down its central axis. The diameter of the wire is to be 0.500 mm and the inside diameter of the tube will be 4.00 cm. The tube is to be filled with a dilute gas in which electrical discharge (breakdown) occurs when the electric field reaches $5.50 \times 10^6$ N/C. Determine the maximum linear charge density on the wire if breakdown of the gas is not to happen. Assume that the tube and the wire are infinitely long.

**Picture the Problem** The electric field of a line charge of infinite length is given by $E_r = \frac{1}{2\pi\in_0}\frac{\lambda}{r}$, where $r$ is the distance from the center of the line of charge and $\lambda$ is the linear charge density of the wire.

The electric field of a line charge of infinite length is given by:

$$E_r = \frac{1}{2\pi\in_0}\frac{\lambda}{r}$$

Because $E_r$ varies inversely with $r$, its maximum value occurs at the surface of the wire where $r = R$, the radius of the wire:

$$E_{\max} = \frac{1}{2\pi\in_0}\frac{\lambda}{R}$$

Solving for $\lambda$ yields:

$$\lambda = 2\pi\in_0 RE_{\max}$$

Substitute numerical values and evaluate $\lambda$:

$$\lambda = 2\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(0.250\,\text{mm})\left(5.50\times10^6\,\frac{\text{N}}{\text{C}}\right) = \boxed{76.5\ \text{nC/m}}$$

**47** ••• In Problem 54, suppose ionizing radiation produces an ion and an electron at a distance of 2.00 cm from the long axis of the central wire of the Geiger tube. Suppose that the central wire is positively charged and has a linear charge density equal to 76.5 pC/m. (*a*) In this case, what will be the electron's speed as it impacts the wire? (*b*) Qualitatively, how will the electron's speed compare to that of the ion's final speed when it impacts the outside cylinder? Explain your reasoning.

**Picture the Problem** Because the inward force on the electron increases as its distance from the wire decreases, we'll need to integrate the net electric force acting on the electron to obtain an expression for its speed as a function of its distance from the wire in the Geiger tube.

(*a*) The force the electron experiences is the radial component of the force on the electron and is the product of its charge and the radial component of the electric field due to the positively charged central wire:

$$F_{e,r} = eE_r$$

The radial electric field due to the charged wire is given by:

$$E_r = \frac{1}{2\pi \in_0}\frac{\lambda}{r}$$

Substituting for $E_r$ yields:

$$F_{e,r} = -\left(\frac{e\lambda}{2\pi \in_0}\right)\frac{1}{r}$$ where the minus sign indicates that the force acting on the electron is radially inward.

Apply Newton's 2nd law to the electron to obtain:

$$-\left(\frac{e\lambda}{2\pi \in_0}\right)\frac{1}{r} = m\frac{dv}{dt} = m\frac{dv}{dt}\frac{dr}{dr} = m\frac{dv}{dr}\frac{dr}{dt} = mv\frac{dv}{dr}$$

Separating variables yields:

$$vdv = -\left(\frac{e\lambda}{2\pi m \in_0}\right)\frac{dr}{r}$$

Express the integral of this equation to obtain:

$$\int_0^{v_f} vdv = -\left(\frac{e\lambda}{2\pi m \in_0}\right)\int_{r_1}^{r_2}\frac{dr}{r}$$ where the lower limit on the left-hand side is zero because the electron is initially at rest.

Integrating yields:

$$\tfrac{1}{2}v_f^2 = -\left(\frac{e\lambda}{2\pi m \in_0}\right)\ln\left(\frac{r_2}{r_1}\right)$$

Solve for $v_f$ to obtain:

$$v_f = \sqrt{\left(\frac{e\lambda}{\pi m \epsilon_0}\right)\ln\left(\frac{r_1}{r_2}\right)}$$

Substitute numerical values and evaluate $v_f$:

$$v_f = \sqrt{\left(\frac{(1.602\times10^{-19}\ \text{C})\left(76.5\ \frac{\text{pC}}{\text{m}}\right)}{\pi(9.109\times10^{-31}\ \text{kg})\left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\right)\ln\left(\frac{0.0200\ \text{m}-0.0025\ \text{m}}{0.250\ \text{mm}}\right)}$$
$$= \boxed{1.46\times10^6\ \text{m/s}}$$

(*b*) The positive ion is accelerated radially outward and will impact the tube instead of the wire. Because of its much larger mass, the impact speed of the ion will be much less than the impact speed of the electron.

**48** •• Show that the electric field due to an infinitely long, uniformly charged thin cylindrical shell of radius $a$ having a surface charge density $\sigma$ is given by the following expressions: $E = 0$ for $0 \le R < a$ and $E_R = \sigma a/(\epsilon_0 R)$ for $R > a$.

**Picture the Problem** From symmetry, the field in the tangential direction must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shell.

Apply Gauss's law to the cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long, uniformly charged cylindrical shell:

$$\oint_S E_n\,dA = \frac{1}{\epsilon_0}Q_{\text{inside}}$$

or

$$2\pi rLE_R = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because no there is no flux through them.

Solve for $E_R$:

$$E_R = \frac{Q_{\text{inside}}}{2\pi rL\epsilon_0} = \frac{2kQ_{\text{inside}}}{Lr}$$

For $r < R$, $Q_{\text{inside}} = 0$ and:

$$E_R(r<R) = \boxed{0}$$

For $r > R$, $Q_{\text{inside}} = \lambda L$ and:

$$E_R(r > R) = \frac{2k\lambda L}{Lr} = \frac{2k\lambda}{r} = \frac{2k(2\pi R\sigma)}{r} = \boxed{\frac{R\sigma}{\epsilon_0 r}}$$

**49** •• A thin cylindrical shell of length 200 m and radius 6.00 cm has a uniform surface charge density of 9.00 nC/m$^2$. (*a*) What is the total charge on the shell? Find the electric field at the following radial distances from the long axis of the cylinder. (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm. (Use the results of Problem 48.)

**Picture the Problem** We can use the definition of surface charge density to find the total charge on the shell. From symmetry, the electric field in the tangential direction must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the uniformly charged cylindrical shell.

(*a*) Using its definition, relate the surface charge density to the total charge on the shell:

$$Q = \sigma A = 2\pi RL\sigma$$

Substitute numerical values and evaluate *Q*:

$$\boldsymbol{Q} = 2\boldsymbol{\pi}(0.0600\,\text{m})(200\,\text{m})(9.00\,\text{nC/m}^2) = \boxed{679\,\text{nC}}$$

(*b*) From Problem 48 we have, for $r = 2.00$ cm:

$$\boldsymbol{E}(2.00\,\text{cm}) = \boxed{0}$$

(*c*) From Problem 48 we have, for $r = 5.90$ cm:

$$E(5.90\,\text{cm}) = \boxed{0}$$

(*d*) From Problem 48 we have, for $r = 6.10$ cm:

$$E(r) = \frac{\sigma R}{\epsilon_0 r}$$

and

$$E(6.10\,\text{cm}) = \frac{(9.00\,\text{nC/m}^2)(0.0600\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.0610\,\text{m})} = \boxed{1.00\,\text{kN/C}}$$

(*e*) From Problem 48 we have, for $r$ = 10.0 cm:

$$E(10.0\,\text{cm}) = \frac{(9.00\,\text{nC/m}^2)(0.0600\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.100\,\text{m})} = \boxed{610\,\text{N/C}}$$

**50** •• An infinitely long non-conducting solid cylinder of radius $a$ has a uniform volume charge density of $\rho_0$. Show that the electric field is given by the following expressions: $E_R = \rho_0 R/(2\in_0)$ for $0 \le R < a$ and $E_R = \rho_0 a^2/(2\in_0 R)$ for $R > a$, where $R$ is the distance from the long axis of the cylinder.

**Picture the Problem** From symmetry, the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

Apply Gauss's law to a cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n\,dA = \frac{1}{\in_0}Q_{\text{inside}}$$

or

$$2\pi rLE_R = \frac{Q_{\text{inside}}}{\in_0}$$

where we've neglected the end areas because there is no flux through them.

Solving for $E_R$ yields:

$$E_R = \frac{Q_{\text{inside}}}{2\pi rL\in_0} = \frac{2kQ_{\text{inside}}}{Lr}$$

Express $\boldsymbol{Q}_{\text{inside}}$ for $r < R$:

$$Q_{\text{inside}} = \rho(r)V = \rho_0(\pi r^2 L)$$

Substitute to obtain:

$$E_R(r<R) = \frac{2k(\pi\rho_0 Lr^2)}{Lr} = \boxed{\frac{\rho_0}{2\in_0}r}$$

or, because $\lambda = \rho\pi R^2$,

$$E_R(r<R) = \boxed{\frac{\lambda}{2\pi\in_0 R^2}r}$$

Express $\boldsymbol{Q}_{\text{inside}}$ for $r > R$:

$$Q_{\text{inside}} = \rho(r)V = \rho_0(\pi R^2 L)$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E_R(r>R)=\frac{2k\left(\pi\rho_0 LR^2\right)}{Lr}=\boxed{\frac{\rho_0 R^2}{2\in_0 r}}$$

or, because $\lambda=\rho\pi R^2$

$$E_R(r>R)=\boxed{\frac{\lambda}{2\pi\in_0 r}}$$

**51 •• [SSM]** A solid cylinder of length 200 m and radius 6.00 cm has a uniform volume charge density of 300 nC/m$^3$. (*a*) What is the total charge of the cylinder? Use the formulas given in Problem 50 to calculate the electric field at a point equidistant from the ends at the following radial distances from the cylindrical axis: (*b*) 2.00 cm, (*c*) 5.90 cm, (*d*) 6.10 cm, and (*e*) 10.0 cm.

**Picture the Problem** We can use the definition of volume charge density to find the total charge on the cylinder. From symmetry, the electric field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the uniformly charged cylinder.

(*a*) Use the definition of volume charge density to express the total charge of the cylinder:

$$Q_{\text{tot}}=\rho V=\rho\left(\pi R^2 L\right)$$

Substitute numerical values to obtain:

$$Q_{\text{tot}}=\pi\left(300\,\text{nC/m}^3\right)(0.0600\,\text{m})^2(200\,\text{m})=\boxed{679\,\text{nC}}$$

(*b*) From Problem 50, for $r<R$, we have:

$$E(r)=\frac{\rho}{2\in_0}r$$

For $r = 2.00$ cm:

$$E(2.00\,\text{cm})=\frac{\left(300\,\text{nC/m}^3\right)(0.0200\,\text{m})}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}=\boxed{339\,\text{N/C}}$$

(*c*) For $r = 5.90$ cm:

$$E(5.90\,\text{cm})=\frac{\left(300\,\text{nC/m}^3\right)(0.0590\,\text{m})}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}=\boxed{1.00\,\text{kN/C}}$$

From Problem 50, for $r > R$, we have: $$E(r) = \frac{\rho R^2}{2\in_0 r}$$

(*d*) For $r = 6.10$ cm:

$$E(6.10\,\text{cm}) = \frac{(300\,\text{nC/m}^3)(0.0600\,\text{m})^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.0610\,\text{m})} = \boxed{1.00\,\text{kN/C}}$$

(*e*) For $r = 10.0$ cm:

$$E(10.0\,\text{cm}) = \frac{(300\,\text{nC/m}^3)(0.0600\,\text{m})^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.100\,\text{m})} = \boxed{610\,\text{N/C}}$$

**52 ••** Consider two infinitely long, coaxial thin cylindrical shells. The inner shell has a radius $a_1$ and has a uniform surface charge density of $\sigma_1$, and the outer shell has a radius $a_2$ and has a uniform surface charge density of $\sigma_2$. (*a*) Use Gauss's law to find expressions for the electric field in the three regions: $0 \le R < a_1$, $a_1 < R < a_2$, and $R > a_2$, where $R$ is the distance from the axis. (*b*) What is the ratio of the surface charge densities $\sigma_2/\sigma_1$ and their relative signs if the electric field is to be zero everywhere outside the largest cylinder? (*c*) For the case in Part (*b*), what would be the electric field between the shells? (*d*) Sketch the electric field lines for the situation in Part (*b*) if $\sigma_1$ is positive.

**Picture the Problem** From symmetry; the field tangent to the surfaces of the shells must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shells.

(*a*) Apply Gauss's law to the cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long, uniformly charged cylindrical shell:

$$\oint_S E_n dA = \frac{1}{\in_0} Q_{\text{inside}} \Rightarrow 2\pi rLE_R = \frac{Q_{\text{inside}}}{\in_0}$$

where we've neglected the end areas because there is no flux through them.

Solving for $E_R$ yields:

$$E_R = \frac{2kQ_{\text{inside}}}{Lr} \qquad (1)$$

For $r < R_1$, $\boldsymbol{Q}_{\text{inside}} = 0$ and: $$E_R(r < R_1) = \boxed{0}$$

Express $Q_{\text{inside}}$ for $R_1 < r < R_2$: $$Q_{\text{inside}} = \sigma_1 A_1 = 2\pi\sigma_1 R_1 L$$

Substitute in equation (1) to obtain:

$$E_R(R_1 < r < R_2) = \frac{2k(2\pi\sigma_1 R_1 L)}{Lr} = \boxed{\frac{\sigma_1 R_1}{\epsilon_0\, r}}$$

Express $Q_{\text{inside}}$ for $r > R_2$:

$$Q_{\text{inside}} = \sigma_1 A_1 + \sigma_2 A_2 = 2\pi\sigma_1 R_1 L + 2\pi\sigma_2 R_2 L$$

Substitute in equation (1) to obtain:

$$E_R(r > R_2) = \frac{2k(2\pi\sigma_1 R_1 L + 2\pi\sigma_2 R_2 L)}{Lr} = \boxed{\frac{\sigma_1 R_1 + \sigma_2 R_2}{\epsilon_0\, r}}$$

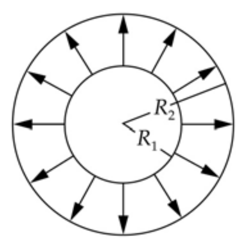
(*b*) Set $E = 0$ for $r > R_2$ to obtain:

$$\frac{\sigma_1 R_1 + \sigma_2 R_2}{\epsilon_0\, r} = 0 \Rightarrow \frac{\sigma_1}{\sigma_2} = \boxed{-\frac{R_2}{R_1}}$$

(*c*) Because the electric field is determined by the charge inside the Gaussian surface, the field under these conditions would be as given above:

$$E_R(R_1 < r < R_2) = \boxed{\frac{\sigma_1 R_1}{\epsilon_0\, r}}$$

(*d*) Because $\sigma_1$ is positive, the field lines are directed as shown to the right:




**53** •• Figure 22-42 shows a portion of an infinitely long, concentric cable in cross section. The inner conductor has a charge of 6.00 nC/m and the outer conductor has no net charge. (*a*) Find the electric field for all values of *R*, where *R* is the perpendicular distance from the common axis of the cylindrical system. (*b*) What are the surface charge densities on the inside and the outside surfaces of the outer conductor?

**Picture the Problem** The electric field is directed radially outward. We can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long, uniformly charged cylindrical shell.

(*a*) Apply Gauss's law to a cylindrical surface of radius *r* and length *L* that is concentric with the inner conductor:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 2\pi r L E_R = \frac{Q_{\text{inside}}}{\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Solving for $E_R$ yields:

$$E_R = \frac{2kQ_{\text{inside}}}{Lr} \qquad (1)$$

For $r < 1.50$ cm, $Q_{\text{inside}} = 0$ and:

$$E_R(r < 1.50\,\text{cm}) = \boxed{0}$$

Letting $R = 1.50$ cm, express $Q_{\text{inside}}$ for 1.50 cm < $r$ < 4.50 cm:

$$Q_{\text{inside}} = \lambda L = 2\pi\sigma R L$$

Substitute in equation (1) to obtain:

$$E_R(1.50\,\text{cm} < r < 4.50\,\text{cm}) = \frac{2k(\lambda L)}{Lr} = \frac{2k\lambda}{r}$$

Substitute numerical values and evaluate $E_n$(1.50 cm < $r$ < 4.50 cm):

$$E_R(1.50\,\text{cm} < r < 4.50\,\text{cm}) = 2(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(6.00\,\text{nC/m})}{r} = \boxed{\frac{(108\,\text{N}\cdot\text{m/C})}{r}}$$

Express $Q_{\text{inside}}$ for 4.50 cm < $r$ < 6.50 cm:

$$Q_{\text{inside}} = 0$$

and

$$E_R(4.50\,\text{cm} < r < 6.50\,\text{cm}) = \boxed{0}$$

Letting $\sigma_2$ represent the charge density on the outer surface, express $Q_{\text{inside}}$ for $r > 6.50$ cm:

$$Q_{\text{inside}} = \sigma_2 A_2 = 2\pi\sigma_2 R_2 L$$

where $R_2 = 6.50$ cm.

Substitute in equation (1) to obtain:

$$E_R(r > R_2) = \frac{2k(2\pi\sigma_2 R_2 L)}{Lr} = \frac{\sigma_2 R_2}{\epsilon_0\, r}$$

In (*b*) we show that $\sigma_2 = 21.22$ nC/m$^2$. Substitute numerical values to obtain:

$$E_R(r > 6.50\,\text{cm}) = \frac{(21.22\,\text{nC/m}^2)(6.50\,\text{cm})}{(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)r} = \boxed{\frac{156\,\text{N}\cdot\text{m/C}}{r}}$$

(*b*) The surface charge densities on the inside and the outside surfaces of the outer conductor are given by:

$$\sigma_{\text{inside}} = \frac{-\lambda}{2\pi R_{\text{inside}}}$$

and

$$\sigma_{\text{outside}} = \frac{\lambda}{2\pi R_{\text{outside}}}$$

Substitute numerical values and evaluate $\sigma_{\text{inside}}$ and $\sigma_{\text{outside}}$:

$$\sigma_{\text{inside}} = \frac{-6.00\,\text{nC/m}}{2\pi(0.0450\,\text{m})} = -21.22\,\text{nC/m}^2$$
$$= \boxed{-21.2\,\text{nC/m}^2}$$

and

$$\sigma_{\text{outside}} = \frac{6.00\,\text{nC/m}}{2\pi(0.0650\,\text{m})}$$
$$= \boxed{14.7\,\text{nC/m}^2}$$

**54** •• An infinitely long non-conducting solid cylinder of radius $a$ has a non-uniform volume charge density. This density varies linearly with $R$, the perpendicular distance from its axis, according to $\rho(R) = \beta R$, where $\beta$ is a constant. (*a*) Show that the linear charge density of the cylinder is given by $\lambda = 2\pi\beta a^3/3$. (*b*) Find expressions for the electric field for $R < a$ and $R > a$.

**Picture the Problem** From symmetry considerations, we can conclude that the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

(*a*) Apply Gauss's law to a cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

or

$$2\pi r L E_n = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow E_R = \frac{Q_{\text{inside}}}{2\pi r L \epsilon_0} \quad (1)$$

where we've neglected the end areas because there is no flux through them.

Express $dQ_{\text{inside}}$ for $\rho(r) = ar$:

$$dQ_{\text{inside}} = \rho(r)dV = ar(2\pi r L)dr$$
$$= 2\pi a r^2 L dr$$

Integrate $dQ_{\text{inside}}$ from $r = 0$ to $R$ to obtain:

$$Q_{\text{inside}} = 2\pi aL\int_0^R r^2 dr = 2\pi aL\left[\frac{r^3}{3}\right]_0^R = \frac{2\pi aL}{3}R^3$$

Divide both sides of this equation by $L$ to obtain an expression for the charge per unit length $\lambda$ of the cylinder:

$$\lambda = \frac{Q_{\text{inside}}}{L} = \boxed{\frac{2\pi aR^3}{3}}$$

($b$) Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

$$E_R(r < R) = \frac{\frac{2\pi aL}{3}r^3}{2\pi \in_0 Lr} = \boxed{\frac{a}{3\in_0}r^2}$$

For $r > R$:

$$Q_{\text{inside}} = \frac{2\pi aL}{3}R^3$$

Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

$$E_R(r > R) = \frac{\frac{2\pi aL}{3}R^3}{2\pi rL \in_0} = \boxed{\frac{aR^3}{3r \in_0}}$$

**55 •• [SSM]** An infinitely long non-conducting solid cylinder of radius $a$ has a non-uniform volume charge density. This density varies with $R$, the perpendicular distance from its axis, according to $\rho(R) = bR^2$, where $b$ is a constant. ($a$) Show that the linear charge density of the cylinder is given by $\lambda = \pi ba^4/2$. ($b$) Find expressions for the electric field for $R < a$ and $R > a$.

**Picture the Problem** From symmetry; the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylinder.

($a$) Apply Gauss's law to a cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n dA = \frac{1}{\in_0}Q_{\text{inside}}$$

or

$$2\pi rLE_n = \frac{Q_{\text{inside}}}{\in_0} \Rightarrow E_R = \frac{Q_{\text{inside}}}{2\pi rL \in_0} \quad (1)$$

where we've neglected the end areas because there is no flux through them.

Express $dQ_{\text{inside}}$ for $\rho(r) = br^2$:

$$dQ_{\text{inside}} = \rho(r)dV = br^2(2\pi rL)dr = 2\pi br^3 Ldr$$

Integrate $dQ_{\text{inside}}$ from $r = 0$ to $R$ to obtain:

$$Q_{\text{inside}} = 2\pi bL\int_0^R r^3 dr = 2\pi bL\left[\frac{r^4}{4}\right]_0^R = \frac{\pi bL}{2}R^4$$

Divide both sides of this equation by $L$ to obtain an expression for the charge per unit length $\lambda$ of the cylinder:

$$\lambda = \frac{Q_{\text{inside}}}{L} = \boxed{\frac{\pi bR^4}{2}}$$

(*b*) Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

$$E_R(r < R) = \frac{\frac{\pi bL}{2}r^4}{2\pi Lr\in_0} = \boxed{\frac{b}{4\in_0}r^3}$$

For $r > R$:

$$Q_{\text{inside}} = \frac{\pi bL}{2}R^4$$

Substitute for $Q_{\text{inside}}$ in equation (1) and simplify to obtain:

$$E_R(r > R) = \frac{\frac{\pi bL}{2}R^4}{2\pi rL\in_0} = \boxed{\frac{bR^4}{4r\in_0}}$$

**56** ••• An infinitely long, non-conducting cylindrical shell of inner radius $a_1$ and outer radius $a_2$ has a uniform volume charge density $\rho$. Find expressions for the electric field everywhere.

**Picture the Problem** From symmetry; the field tangent to the surface of the cylinder must vanish. We can construct a Gaussian surface in the shape of a cylinder of radius $r$ and length $L$ and apply Gauss's law to find the electric field as a function of the distance from the centerline of the infinitely long nonconducting cylindrical shell.

Apply Gauss's law to a cylindrical surface of radius $r$ and length $L$ that is concentric with the infinitely long nonconducting cylindrical shell:

$$\oint_S E_n dA = \frac{1}{\in_0}Q_{\text{inside}}$$

or

$$2\pi rLE_n = \frac{Q_{\text{inside}}}{\in_0} \Rightarrow E_R = \frac{Q_{\text{inside}}}{2\pi rL\in_0}$$

where we've neglected the end areas because no flux crosses them.

For $r < R_1$, $Q_{\text{inside}} = 0$:

$$E_R(r < R_1) = \boxed{0}$$

Express $Q_{\text{inside}}$ for $R_1 < r < R_2$:

$$Q_{\text{inside}} = \rho V = \rho\pi r^2 L - \rho\pi a^2 L = \rho\pi L\left(r^2 - R_1^2\right)$$

Substitute for $Q_{\text{inside}}$ and simplify to obtain:

$$E_R(R_1 < r < R_2) = \frac{\rho\pi L\left(r^2 - R_1^2\right)}{2\pi \in_0 Lr} = \boxed{\frac{\rho\left(r^2 - R_1^2\right)}{2 \in_0 r}}$$

Express $Q_{\text{inside}}$ for $r > R_2$:

$$Q_{\text{inside}} = \rho V = \rho\pi b^2 L - \rho\pi a^2 L = \rho\pi L\left(R_2^2 - R_1^2\right)$$

Substitute for $Q_{\text{inside}}$ and simplify to obtain:

$$E_R(r > b) = \frac{\rho\pi L\left(R_2^2 - R_1^2\right)}{2\pi \in_0 rL} = \boxed{\frac{\rho\left(R_2^2 - R_1^2\right)}{2 \in_0 r}}$$

**57 ••• [SSM] T**he inner cylinder of Figure 22-42 is made of non-conducting material and has a volume charge distribution given by $\rho(R) = C/R$, where $C = 200\ \text{nC/m}^2$. The outer cylinder is metallic, and both cylinders are infinitely long. (*a*) Find the charge per unit length (that is, the linear charge density) on the inner cylinder. (*b*) Calculate the electric field for all values of *R*.

**Picture the Problem** We can integrate the density function over the radius of the inner cylinder to find the charge on it and then calculate the linear charge density from its definition. To find the electric field for all values of *r* we can construct a Gaussian surface in the shape of a cylinder of radius *r* and length *L* and apply Gauss's law to each region of the cable to find the electric field as a function of the distance from its centerline.

(*a*) Find the charge $Q_{\text{inner}}$ on the inner cylinder:

$$Q_{\text{inner}} = \int_0^R \rho(r)dV = \int_0^R \frac{C}{r} 2\pi rL dr = 2\pi CL\int_0^R dr = 2\pi CLR$$

Relate this charge to the linear charge density:

$$\lambda_{\text{inner}} = \frac{Q_{\text{inner}}}{L} = \frac{2\pi CLR}{L} = 2\pi CR$$

Substitute numerical values and evaluate $\lambda_{\text{inner}}$:

$$\lambda_{\text{inner}} = 2\pi(200\,\text{nC/m})(0.0150\,\text{m}) = \boxed{18.8\,\text{nC/m}}$$

(*b*) Apply Gauss's law to a cylindrical surface of radius *r* and length *L* that is concentric with the infinitely long nonconducting cylinder:

$$\oint_S E_n\,dA = \frac{1}{\epsilon_0}Q_{\text{inside}}$$

or

$$2\pi rLE_n = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow E_R = \frac{Q_{\text{inside}}}{2\pi rL\,\epsilon_0}$$

where we've neglected the end areas because there is no flux through them.

Substitute to obtain, for $r < 1.50$ cm:

$$E_R(r < 1.50\,\text{cm}) = \frac{2\pi CLr}{2\pi\,\epsilon_0\,Lr} = \frac{C}{\epsilon_0}$$

Substitute numerical values and evaluate $E_R(r < 1.50\text{ cm})$:

$$E_R(r < 1.50\,\text{cm}) = \frac{200\,\text{nC/m}^2}{8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{22.6\,\text{kN/C}}$$

Express $Q_{\text{inside}}$ for $1.50\text{ cm} < r < 4.50\text{ cm}$:

$$Q_{\text{inside}} = 2\pi CLR$$

Substitute to obtain, for $1.50\text{ cm} < r < 4.50\text{ cm}$:

$$E_R(1.50\,\text{cm} < r < 4.50\,\text{cm}) = \frac{2C\pi RL}{2\pi\,\epsilon_0\,rL} = \frac{CR}{\epsilon_0\,r}$$

where $R = 1.50$ cm.

Substitute numerical values and evaluate $E_n(1.50\text{ cm} < r < 4.50\text{ cm})$:

$$E_R(1.50\,\text{cm} < r < 4.50\,\text{cm}) = \frac{(200\,\text{nC/m}^2)(0.0150\,\text{m})}{(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)r} = \boxed{\frac{339\,\text{N}\cdot\text{m/C}}{r}}$$

Because the outer cylindrical shell is a conductor:

$$E_R(4.50\,\text{cm} < r < 6.50\,\text{cm}) = \boxed{0}$$

For $r > 6.50$ cm, $Q_{\text{inside}} = 2\pi CLR$ and:

$$E_R(r > 6.50\,\text{cm}) = \boxed{\frac{339\,\text{N}\cdot\text{m/C}}{r}}$$

## Electric Charge and Field at Conductor Surfaces

**58** • An uncharged penny is in a region that has a uniform electric field of magnitude 1.60 kN/C directed perpendicular to its faces. (*a*) Find the charge density on each face of the penny, assuming the faces are planes. (*b*) If the radius of the penny is 1.00 cm, find the total charge on one face.

**Picture the Problem** Because the penny is in an external electric field, it will have charges of opposite signs induced on its faces. The induced charge $\sigma$ is related to the electric field by $E = \sigma/\epsilon_0$. Once we know $\sigma$, we can use the definition of surface charge density to find the total charge on one face of the penny.

(*a*) Relate the electric field to the charge density on each face of the penny:

$$E = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = \epsilon_0 E$$

Substitute numerical values and evaluate $\sigma$:

$$\sigma = \left(8.854\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(1.60\,\mathrm{kN/C}\right) = 14.17\,\mathrm{nC/m^2} = \boxed{14.2\,\mathrm{nC/m^2}}$$

(*b*) Use the definition of surface charge density to obtain:

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi r^2} \Rightarrow Q = \sigma\pi r^2$$

Substitute numerical values and evaluate $Q$:

$$Q = \pi\left(14.17\,\mathrm{nC/m^2}\right)\left(0.0100\,\mathrm{m}\right)^2 = \boxed{4.45\,\mathrm{pC}}$$

**59** • A thin metal slab has a net charge of zero and has square faces that have 12-cm-long sides. It is in a region that has a uniform electric field that is perpendicular to its faces. The total charge induced on one of the faces is 1.2 nC. What is the magnitude of the electric field?

**Picture the Problem** Because the metal slab is in an external electric field, it will have charges of opposite signs induced on its faces. The induced charge $\sigma$ is related to the electric field by $E = \sigma/\epsilon_0$.

Relate the magnitude of the electric field to the charge density on the metal slab:

$$E = \frac{\sigma}{\epsilon_0}$$

Use its definition to express $\sigma$:

$$\sigma = \frac{Q}{A} = \frac{Q}{L^2}$$

Substitute for $\sigma$ to obtain:

$$E = \frac{Q}{L^2 \epsilon_0}$$

Substitute numerical values and evaluate $E$:

$$E = \frac{1.2\,\text{nC}}{(0.12\,\text{m})^2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)} = \boxed{9.4\,\text{kN/C}}$$

**60** •• A charge of -6.00 nC is uniformly distributed on a thin square sheet of non-conducting material of edge length 20.0 cm. (*a*) What is the surface charge density of the sheet? (*b*) What are the magnitude and direction of the electric field next to the sheet and proximate to the center of the sheet?

**Picture the Problem** We can apply its definition to find the surface charge density of the nonconducting material and calculate the electric field at either of its surfaces from $\sigma/(2\epsilon_0)$.

(*a*) Use its definition to find $\sigma$:

$$\sigma = \frac{Q}{A} = \frac{-6.00\,\text{nC}}{(0.200\,\text{m})^2} = \boxed{-150\,\text{nC/m}^2}$$

(*b*) The magnitude of the electric field just outside the surface of the sheet on the side that is charged is given by:

$$|E| = \frac{\sigma}{2\epsilon_0} = \left|\frac{-150\,\text{nC/m}^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\right| = \boxed{8.47\,\text{kN/C}}$$

The direction of the field on the side of the sheet that is charged is the direction of the electric force acting on a test charge. Because the surface is negatively charged, this force and, hence, the electric field, is directed toward the surface.

Because the sheet is constructed from non-conducting material, no charge is induced on the second surface of the sheet and there is, therefore, no electric field just outside the sheet surface on this side.

**61** • A conducting spherical shell that has zero net charge has an inner radius $R_1$ and an outer radius $R_2$. A positive point charge $q$ is placed at the center of the shell. (*a*) Use Gauss's law and the properties of conductors in electrostatic equilibrium to find the electric field in the three regions: $0 \le r < R_1$, $R_1 < r < R_2$, and $r > R_2$, where $r$ is the distance from the center. (*b*) Draw the electric field lines in all three regions. (*c*) Find the charge density on the inner surface ($r = R_1$) and on the outer surface ($r = R_2$) of the shell.

**Picture the Problem** We can construct a Gaussian surface in the shape of a sphere of radius $r$ with the same center as the shell and apply Gauss's law to find

the electric field as a function of the distance from this point. The inner and outer surfaces of the shell will have charges induced on them by the charge $q$ at the center of the shell.

(*a*) Apply Gauss's law to a spherical surface of radius $r$ that is concentric with the point charge:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solving for $E_r$ yields:

$$E_r = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} \qquad (1)$$

For $r < R_1$, $Q_{\text{inside}} = q$. Substitute in equation (1) and simplify to obtain:

$$E_r(r < R_1) = \frac{q}{4\pi r^2 \epsilon_0} = \boxed{\frac{kq}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for $R_1 < r < R_2$:

$$Q_{\text{inside}} = 0$$

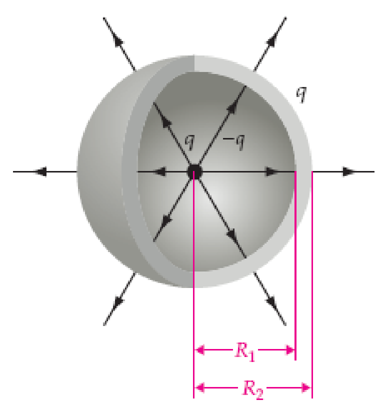
and

$$E_r(R_1 < r < R_2) = \boxed{0}$$

For $r > R_2$, $Q_{\text{inside}} = q$. Substitute in equation (1) and simplify to obtain:

$$E_r(r > R_2) = \frac{q}{4\pi r^2 \epsilon_0} = \boxed{\frac{kq}{r^2}}$$

(*b*) The electric field lines are shown in the diagram to the right:




(*c*) A charge $-q$ is induced on the inner surface. Use the definition of surface charge density to obtain:

$$\sigma_{\text{inner}} = \boxed{-\frac{q}{4\pi R_1^2}}$$

A charge $q$ is induced on the outer surface. Use the definition of surface charge density to obtain:

$$\sigma_{\text{outer}} = \boxed{\frac{q}{4\pi R_2^2}}$$

**62** •• The electric field just above the surface of Earth has been measured to typically be 150 N/C pointing downward. (*a*) What is the sign of the net charge on Earth's surface under typical conditions? (*b*)What is the total charge on Earth's surface implied by this measurement?

**Picture the Problem** We can construct a spherical Gaussian surface at the surface of Earth (we'll assume Earth is a sphere) and apply Gauss's law to relate the electric field to its total charge.

(*a*) Because the direction of an electric field is the direction of the force acting on a positively charged object, the net charge on Earth's surface must be negative.

(*b*)Apply Gauss's law to a spherical surface of radius $R_E$ that is concentric with Earth:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{inside} \Rightarrow 4\pi R_E^2 E_n = \frac{Q_{inside}}{\epsilon_0}$$

Solve for $Q_{inside} = Q_{Earth}$ to obtain:

$$Q_{Earth} = 4\pi \epsilon_0 R_E^2 E_n = \frac{R_E^2 E_n}{k}$$

Substitute numerical values and evaluate $Q_{Earth}$:

$$\boldsymbol{Q}_{Earth} = \frac{\left(6.37\times10^6\ \text{m}\right)^2 (150\ \text{N/C})}{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2} = \boxed{677\ \text{kC}}$$

**63** •• **[SSM]** A positive point charge of 2.5 $\mu$C is at the center of a conducting spherical shell that has a net charge of zero, an inner radius equal to 60 cm, and an outer radius equal to 90 cm. (*a*) Find the charge densities on the inner and outer surfaces of the shell and the total charge on each surface. (*b*) Find the electric field everywhere. (*c*) Repeat Part (*a*) and Part (*b*) with a net charge of +3.5 $\mu$C placed on the shell.

**Picture the Problem** Let the inner and outer radii of the uncharged spherical conducting shell be $R_1$ and $R_2$ and $q$ represent the positive point charge at the center of the shell. The positive point charge at the center will induce a negative charge on the inner surface of the shell and, because the shell is uncharged, an equal positive charge will be induced on its outer surface. To solve Part (*b*), we can construct a Gaussian surface in the shape of a sphere of radius $r$ with the same center as the shell and apply Gauss's law to find the electric field as a function of the distance from this point. In Part (*c*) we can use a similar strategy with the additional charge placed on the shell.

(*a*) Express the charge density on the inner surface:

$$\sigma_{inner} = \frac{q_{inner}}{A}$$

Express the relationship between the positive point charge $q$ and the charge induced on the inner surface $q_{\text{inner}}$:

$$q + q_{\text{inner}} = 0 \Rightarrow q_{\text{inner}} = -q$$

Substitute for $q_{\text{inner}}$ and $A$ to obtain:

$$\sigma_{\text{inner}} = \frac{-q}{4\pi R_1^2}$$

Substitute numerical values and evaluate $\sigma_{\text{inner}}$:

$$\sigma_{\text{inner}} = \frac{-2.5\,\mu\text{C}}{4\pi(0.60\,\text{m})^2} = \boxed{-0.55\,\mu\text{C/m}^2}$$

Express the charge density on the outer surface:

$$\sigma_{\text{outer}} = \frac{q_{\text{outer}}}{A}$$

Because the spherical shell is uncharged:

$$q_{\text{outer}} + q_{\text{inner}} = 0$$

Substitute for $q_{\text{outer}}$ to obtain:

$$\sigma_{\text{outer}} = \frac{-q_{\text{inner}}}{4\pi R_2^2}$$

Substitute numerical values and evaluate $\sigma_{\text{outer}}$:

$$\sigma_{\text{outer}} = \frac{2.5\,\mu\text{C}}{4\pi(0.90\,\text{m})^2} = \boxed{0.25\,\mu\text{C/m}^2}$$

(*b*) Apply Gauss's law to a spherical surface of radius $r$ that is concentric with the point charge:

$$\oint_S E_n\,dA = \frac{1}{\epsilon_0} Q_{\text{inside}} \Rightarrow 4\pi r^2 E_r = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Solve for $E_r$:

$$E_r = \frac{Q_{\text{inside}}}{4\pi r^2 \epsilon_0} \qquad (1)$$

For $r < R_1 = 60$ cm, $Q_{\text{inside}} = q$. Substitute in equation (1) and evaluate $E_r(r < 60\text{ cm})$ to obtain:

$$E_r(r < 60\,\text{cm}) = \frac{q}{4\pi r^2 \epsilon_0} = \frac{kq}{r^2} = \frac{(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(2.5\,\mu\text{C})}{r^2}$$

$$= \boxed{(2.3\times10^4\,\text{N}\cdot\text{m}^2/\text{C})\frac{1}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for 60 cm < $r$ < 90 cm:

$$Q_{\text{inside}} = 0$$

and

$$E_r(60\,\text{cm} < r < 90\,\text{cm}) = \boxed{0}$$

For $r$ > 90 cm, the net charge inside the Gaussian surface is $q$ and:

$$E_r(r > 90\,\text{cm}) = \frac{kq}{r^2} = \boxed{\left(2.3\times10^4\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

($c$) Because $E$ = 0 in the conductor:

$$q_{\text{inner}} = -2.5\ \mu\text{C}$$

and

$$\sigma_{\text{inner}} = \boxed{-0.55\ \mu\text{C/m}^2}$$ as before.

Express the relationship between the charges on the inner and outer surfaces of the spherical shell:

$$q_{\text{outer}} + q_{\text{inner}} = 3.5\ \mu\text{C}$$

and

$$q_{\text{outer}} = 3.5\ \mu\text{C} - q_{\text{inner}} = 6.0\ \mu\text{C}$$

$\sigma_{\text{outer}}$ is now given by:

$$\sigma_{\text{outer}} = \frac{6.0\ \mu\text{C}}{4\pi(0.90\,\text{m})^2} = \boxed{0.59\ \mu\text{C/m}^2}$$

For $r < R_1$ = 60 cm, $Q_{\text{inside}} = q$ and $E_r(r$ < 60 cm) is as it was in ($a$):

$$E_r(r < 60\ \text{cm}) = \boxed{\left(2.3\times10^4\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

Because the spherical shell is a conductor, a charge $-q$ will be induced on its inner surface. Hence, for 60 cm < $r$ < 90 cm:

$$Q_{\text{inside}} = 0$$

and

$$E_r(60\,\text{cm} < r < 90\,\text{cm}) = \boxed{0}$$

For $r$ > 0.90 m, the net charge inside the Gaussian surface is 6.0 $\mu$C and:

$$E_r(r > 90\,\text{cm}) = \frac{kq}{r^2} = \left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(6.0\ \mu\text{C})\frac{1}{r^2} = \boxed{\left(5.4\times10^4\ \text{N}\cdot\text{m}^2/\text{C}\right)\frac{1}{r^2}}$$

**64 ••** If the magnitude of an electric field in air is as great as $3.0 \times 10^6$ N/C, the air becomes ionized and begins to conduct electricity. This phenomenon is called *dielectric breakdown*. A charge of 18 $\mu$C is to be placed on a conducting

sphere. What is the minimum radius of a sphere that can hold this charge without breakdown?

**Picture the Problem** From Gauss's law we know that the electric field at the surface of the charged sphere is given by $E = kQ/R^2$ where $Q$ is the charge on the sphere and $R$ is its radius. The minimum radius for dielectric breakdown corresponds to the maximum electric field at the surface of the sphere.

Use Gauss's law to express the electric field at the surface of the charged sphere:

$$E = \frac{kQ}{R^2}$$

Express the relationship between $E$ and $R$ for dielectric breakdown:

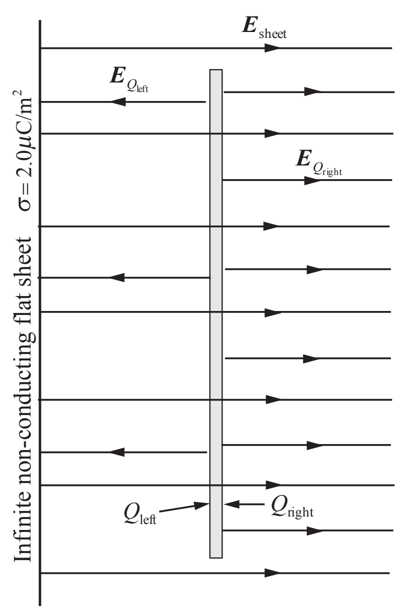
$$E_{max} = \frac{kQ}{R_{min}^2} \Rightarrow R_{min} = \sqrt{\frac{kQ}{E_{max}}}$$

Substitute numerical values and evaluate $R_{min}$:

$$R_{min} = \sqrt{\frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(18\ \mu\text{C})}{3.0\times10^6\ \text{N/C}}} = \boxed{23\,\text{cm}}$$

**65 ••• [SSM]** A square conducting slab carries a net charge of 80 $\mu$C. The dimensions of the slab are 1.0 cm × 5.0 m × 5.0 m. To the left of the slab is an infinite non-conducting flat sheet with charge density 2.0 $\mu$C/m$^2$. The faces of the slab are parallel to the sheet. (*a*) Find the charge on each face of the slab. (Assume that on each face of the slab the surface charge is uniformly distributed, and that the amount of charge on the edges of the slab is negligible.) (*b*) Find the electric field just to the left of the slab and just to the right of the slab.

**Picture the Problem** (*a*) We can use the fact that the net charge on the conducting slab is the sum of the charges $Q_{left}$ and $Q_{right}$ on its left and right surfaces to obtain a linear equation relating these charges. Because the electric field is zero inside the slab, we can obtain a second linear equation in $Q_{left}$ and $Q_{right}$ that we can solve simultaneously with the first equation to find $Q_{left}$ and $Q_{right}$. (*b*) We can find the electric field on each side of the slab by adding the fields due to the charges on the surfaces of the slab and the field due to the plane of charge.



(*a*) The net charge on the conducting slab is the sum of the charges on the surfaces to the left and to the right:

$$Q_{\text{left}} + Q_{\text{right}} = 80\ \mu\text{C} \qquad (1)$$

Because the electric field is equal to zero inside the slab:

$$\frac{\sigma_{\text{sheet}}}{2\in_0} + \frac{\sigma_{\text{left}}}{2\in_0} - \frac{\sigma_{\text{right}}}{2\in_0} = 0$$

Letting $A$ represent the area of the charged surfaces of the slab and substituting for $\sigma_{\text{left}}$ and $\sigma_{\text{rightt}}$ yields:

$$\frac{\sigma_{\text{sheet}}}{2\in_0} + \frac{Q_{\text{left}}}{2A\in_0} - \frac{Q_{\text{right}}}{2A\in_0} = 0$$

Simplifying to obtain:

$$A\sigma_{\text{sheet}} + Q_{\text{left}} - Q_{\text{right}} = 0$$

Substituting numerical values yields:

$$(5.0\ \text{m})^2\left(2.0\ \frac{\mu\text{C}}{\text{m}^2}\right) + Q_{\text{left}} - Q_{\text{right}} = 0$$

or

$$Q_{\text{left}} - Q_{\text{right}} = -50\ \mu\text{C} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$Q_{\text{left}} = \boxed{15\ \mu\text{C}} \text{ and } Q_{\text{right}} = \boxed{65\ \mu\text{C}}$$

(*b*) Express the total field just to the left of the slab:

$$\vec{E}_{\substack{\text{left of}\\ \text{the slab}}} = \vec{E}_{\text{sheet}} + \vec{E}_{Q_{\text{left}}} + \vec{E}_{Q_{\text{right}}} = \frac{\sigma_{\text{sheet}}}{2\in_0}\hat{r} - \frac{\sigma_{\text{left}}}{2\in_0}\hat{r} - \frac{\sigma_{\text{right}}}{2\in_0}\hat{r}$$

where $\hat{r}$ is a unit vector pointing away from the slab.

Substituting for $\sigma_{\text{left}}$ and $\sigma_{\text{right}}$ and simplifying yields:

$$\vec{E}_{\substack{\text{left of}\\ \text{the slab}}} = \frac{\sigma_{\text{sheet}}}{2\in_0}\hat{r} - \frac{Q_{\text{left}}}{2A\in_0}\hat{r} - \frac{Q_{\text{right}}}{2A\in_0}\hat{r} = \frac{A\sigma_{\text{sheet}} - (Q_{\text{left}} + Q_{\text{right}})}{2A\in_0}\hat{r}$$

Substitute numerical values and evaluate $\vec{E}_{\substack{\text{left of}\\ \text{the slab}}}$:

$$\vec{E}_{\substack{\text{left of}\\ \text{the slab}}} = \frac{(5.0\ \text{m})^2(2.0\ \mu\text{C/m}^2) - (15\ \mu\text{C} + 65\mu\text{C})}{2(5.0\ \text{m})^2(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}\hat{r} = \boxed{(-68\,\text{kN/C})\,\hat{r}}$$

Express the total field just to the right of the slab:

$$\vec{E}_{\substack{\text{right of}\\\text{the slab}}} = \vec{E}_{\text{sheet}} + \vec{E}_{Q_{\text{left}}} + \vec{E}_{\substack{\text{right surface}\\\text{of the slab}}} = \frac{\sigma_{\text{sheet}}}{2\in_0}\hat{r} + \frac{\sigma_{\text{left}}}{2\in_0}\hat{r} + \frac{\sigma_{\text{right}}}{2\in_0}\hat{r}$$

Substituting for $\sigma_{\text{left}}$ and $\sigma_{\text{right}}$ and simplifying yields:

$$\vec{E}_{\substack{\text{right of}\\\text{the slab}}} = \frac{\sigma_{\text{sheet}}}{2\in_0}\hat{r} + \frac{Q_{\text{left}}}{2A\in_0}\hat{r} + \frac{Q_{\text{right}}}{2A\in_0}\hat{r} = \frac{A\sigma_{\text{sheet}} + \left(Q_{\text{right}} + Q_{\text{left}}\right)}{2A\in_0}\hat{r}$$

Substitute numerical values and evaluate $\vec{E}_{\substack{\text{right of}\\\text{the slab}}}$:

$$\vec{E}_{\substack{\text{right of}\\\text{the slab}}} = \frac{(5.0\text{ m})^2\left(2.0\,\mu\text{C/m}^2\right) + \left(65\,\mu\text{C} + 15\,\mu\text{C}\right)}{2(5.0\text{ m})^2\left(8.854\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2\right)}\hat{r} = \boxed{(0.29\,\text{MN/C})\,\hat{r}}$$

## General Problems

**66** •• Consider the concentric metal sphere and spherical shells that are shown in Figure 22-43. The innermost is a solid sphere that has a radius $R_1$. A spherical shell surrounds the sphere and has an inner radius $R_2$ and an outer radius $R_3$. The sphere and the shell are both surrounded by a second spherical shell that has an inner radius $R_4$ and an outer radius $R_5$. None of these three objects initially have a net charge. Then, a negative charge $-Q_0$ is placed on the inner sphere and a positive charge $+Q_0$ is placed on the outermost shell. (*a*) After the charges have reached equilibrium, what will be the direction of the electric field between the inner sphere and the middle shell? (*b*) What will be the charge on the inner surface of the middle shell? (*c*) What will be the charge on the outer surface of the middle shell? (*d*) What will be the charge on the inner surface of the outermost shell? (*e*) What will be the charge on the outer surface of the outermost shell? (*f*) Plot $E$ as a function of $r$ for all values of $r$.

**Determine the Concept** We can determine the direction of the electric field between spheres I and II by imagining a test charge placed between the spheres and determining the direction of the force acting on it. We can determine the amount and sign of the charge on each sphere by realizing that the charge on a given surface induces a charge of the same magnitude but opposite sign on the next surface of larger radius.

(*a*) The charge placed on sphere III has no bearing on the electric field between spheres I and II. The field in this region will be in the direction of the force exerted on a test charge placed between the spheres. Because the charge at the center is negative, $\boxed{\text{the field will point toward the center.}}$

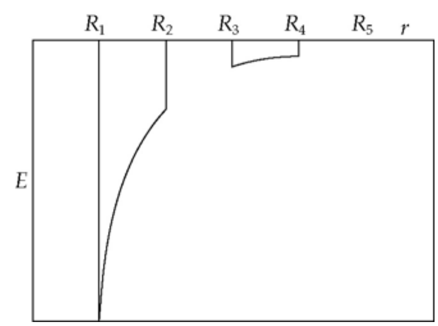
(*b*) The charge on sphere I ($-Q_0$) will induce a charge of the same magnitude but opposite sign on sphere II: $\boxed{+Q_0}$

(*c*) The induction of charge $+Q_0$ on the inner surface of sphere II will leave its outer surface with a charge of the same magnitude but opposite sign: $\boxed{-Q_0}$

(*d*) The presence of charge $-Q_0$ on the outer surface of sphere II will induce a charge of the same magnitude but opposite sign on the inner surface of sphere III: $\boxed{+Q_0}$

(*e*) The presence of charge $+Q_0$ on the inner surface of sphere III will leave the outer surface of sphere III neutral: $\boxed{0}$

(*f*) A graph of $E$ as a function of $r$ is shown to the right:




**67 •• [SSM]** A large, flat, nonconducting, non-uniformly charged surface lies in the $x = 0$ plane. At the origin, the surface charge density is $+3.10\ \mu\text{C/m}^2$. A small distance away from the surface on the positive $x$ axis, the $x$ component of the electric field is $4.65 \times 10^5$ N/C. What is $E_x$ a small distance away from the surface on the negative $x$ axis?

**Picture the Problem** Because the difference between the field just to the right of the surface $\boldsymbol{E}_{x,\text{pos}}$ and the field just to the left of the surface $\boldsymbol{E}_{x,\text{neg}}$ is the field due to the nonuniform surface charge, we can express $\boldsymbol{E}_{x,\text{neg}}$ as the difference between $\boldsymbol{E}_{x,\text{pos}}$ and $\sigma/\epsilon_0$.

Express the electric field just to the left of the origin in terms of $\boldsymbol{E}_{x,\text{pos}}$ and $\sigma/\epsilon_0$:

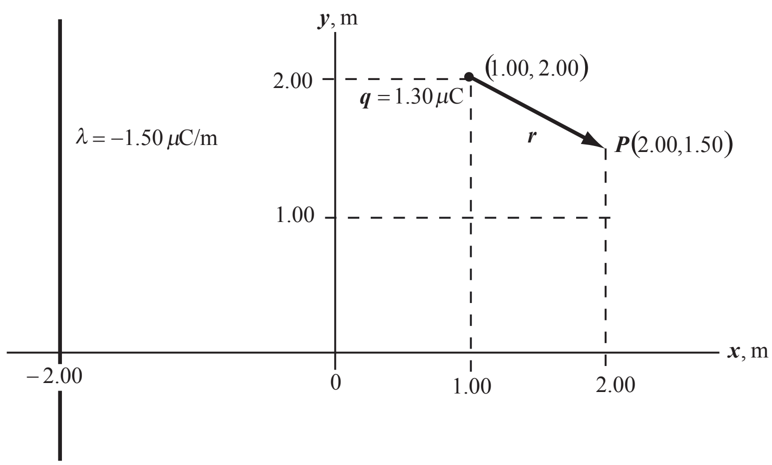
$$\boldsymbol{E}_{x,\text{neg}} = \boldsymbol{E}_{x,\text{pos}} - \frac{\boldsymbol{\sigma}}{\epsilon_0}$$

Substitute numerical values and evaluate $E_{x,\text{neg}}$:

$$E_{x,\text{neg}} = 4.65\times10^{5}\ \text{N/C} - \frac{3.10\,\mu\text{C/m}^2}{8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{115\ \text{kN/C}}$$

**68** •• An infinitely long line charge that has a uniform linear charge density equal to –1.50 $\mu$C/m lies parallel to the $y$ axis at $x$ = –2.00 m. A positive point charge that has a magnitude equal to 1.30 $\mu$C is located at $x$ = 1.00 m, $y$ = 2.00 m. Find the electric field at $x$ = 2.00 m, $y$ = 1.50 m.

**Picture the Problem** Let $P$ denote the point of interest at (2.00 m, 1.50 m). The electric field at $P$ is the sum of the electric fields due to the infinite line charge and the point charge.




Express the resultant electric field at $P$: $\vec{E} = \vec{E}_\lambda + \vec{E}_q$

Find the field at $P$ due the infinite line charge:

$$\vec{E}_\lambda = \frac{2k\lambda}{r}\hat{r} = \frac{2\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(-1.50\,\mu\text{C/m}\right)}{4.00\,\text{m}}\hat{i} = \left(-6.741\,\text{kN/C}\right)\hat{i}$$

Express the field at $P$ due the point charge: $\vec{E}_q = \frac{kq}{r^2}\hat{r}$

Referring to the diagram above, determine $r$ and $\hat{r}$: $r = 1.118\,\text{m}$ and $\hat{r} = 0.8944\hat{i} - 0.4472\hat{j}$

Substitute and evaluate $\vec{E}_q(2.00\,\text{m},1.50\,\text{m})$:

$$\vec{E}_q(2.00\,\text{m},1.50\,\text{m}) = \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(1.30\,\mu\text{C})}{(1.118\,\text{m})^2}\left(0.8944\hat{\boldsymbol{i}} - 0.4472\hat{\boldsymbol{j}}\right)$$
$$= (9.348\,\text{kN/C})\left(0.8944\hat{\boldsymbol{i}} - 0.4472\hat{\boldsymbol{j}}\right)$$
$$= (8.361\,\text{kN/C})\hat{\boldsymbol{i}} - (4.180\,\text{kN/C})\hat{\boldsymbol{j}}$$

Substitute and simplify to obtain:

$$\vec{E}(2.00\,\text{m},1.50\,\text{m}) = (-6.741\,\text{kN/C})\hat{\boldsymbol{i}} + (8.361\,\text{kN/C})\hat{\boldsymbol{i}} - (4.180\,\text{kN/C})\hat{\boldsymbol{j}}$$
$$= \boxed{(1.62\,\text{kN/C})\hat{\boldsymbol{i}} - (4.18\,\text{kN/C})\hat{\boldsymbol{j}}}$$

**69 •• [SSM]** A thin, non-conducting, uniformly charged spherical shell of radius *R* (Figure 22-44*a*) has a total positive charge of *Q*. A small circular plug is removed from the surface. (*a*) What is the magnitude and direction of the electric field at the center of the hole? (*b*) The plug is now put back in the hole (Figure 22-44*b*). Using the result of Part (*a)*, find the electric force acting on the plug. (*c*) Using the magnitude of the force, calculate the "electrostatic pressure" (force/unit area) that tends to expand the sphere.

**Picture the Problem** If the patch is small enough, the field at the center of the patch comes from two contributions. We can view the field in the hole as the sum of the field from a uniform spherical shell of charge *Q* plus the field due to a small patch with surface charge density equal but opposite to that of the patch cut out.

(*a*) Express the magnitude of the electric field at the center of the hole:

$$E = E_{\substack{\text{spherical}\\\text{shell}}} + E_{\text{hole}}$$

Apply Gauss's law to a spherical gaussian surface just outside the given sphere:

$$E_{\substack{\text{spherical}\\\text{shell}}}\left(4\pi r^2\right) = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for $\boldsymbol{E}_{\substack{\text{spherical}\\\text{shell}}}$ to obtain:

$$E_{\substack{\text{spherical}\\\text{shell}}} = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field due to the small hole (small enough so that we can treat it as a plane surface) is:

$$E_{\text{hole}} = \frac{-\sigma}{2\epsilon_0}$$

Substitute and simplify to obtain:

$$E = \frac{Q}{4\pi \in_0 r^2} + \frac{-\sigma}{2\in_0} = \frac{Q}{4\pi \in_0 r^2} - \frac{Q}{2\in_0 \left(4\pi r^2\right)} = \boxed{\frac{Q}{8\pi \in_0 r^2}\ \text{radially outward}}$$

(*b*) Express the force on the patch:

$$F = qE$$

where $q$ is the charge on the patch.

Assuming that the patch has radius *a,* express the proportion between its charge and that of the spherical shell:

$$\frac{q}{\pi a^2} = \frac{Q}{4\pi r^2} \text{ or } q = \frac{a^2}{4r^2}Q$$

Substitute for $q$ and $E$ in the expression for $F$ to obtain:

$$F = \left(\frac{a^2}{4r^2}Q\right)\left(\frac{Q}{8\pi \in_0 r^2}\right) = \boxed{\frac{Q^2 a^2}{32\pi \in_0 r^4}\ \text{radially outward}}$$

(*c*) The pressure is the force exerted on the patch divided by the area of the patch:

$$P = \frac{\dfrac{Q^2 a^2}{32\pi \in_0 r^4}}{\pi a^2} = \boxed{\frac{Q^2}{32\pi^2 \in_0 r^4}}$$

**70 ••** An infinite thin sheet in the $y = 0$ plane has a uniform surface charge density $\sigma_1 = +65\ \text{nC/m}^2$. A second infinite thin sheet has a uniform charge density $\sigma_2 = +45\ \text{nC/m}^2$ and intersects the $y = 0$ plane at the $z$ axis and makes an angle of 30° with the $xz$ plane, as shown in Figure 22-45. Find the electric field at (*a*) $x = 6.0$ m, $y = 2.0$ m and (*b*) $x = 6.0$ m, $y = 5.0$ m.

**Picture the Problem** Let the numeral 1 refer to the plane with charge density $\sigma_1$ and the numeral 2 to the plane with charge density $\sigma_2$. We can find the electric field at the two points of interest by adding the electric fields due to the charge distributions of the two infinite planes.

Express the electric field at any point in space due to the charge distributions on the two planes:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \qquad (1)$$

(*a*) Express the electric field at (6.0 m, 2.0 m) due to plane 1:

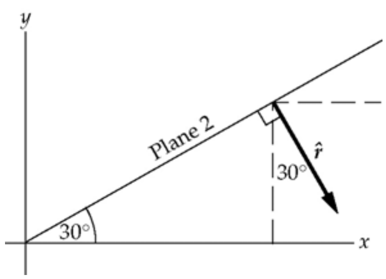
$$\vec{E}_1(6.0\,\text{m},2.0\,\text{m}) = \frac{\sigma_1}{2\epsilon_0}\hat{j} = \frac{+65\,\text{nC/m}^2}{2(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{j} = (3.67\,\text{kN/C})\hat{j}$$

Express the electric field at (6.0 m, 2.0 m) due to plane 2:

$$\vec{E}_2(6.0\,\text{m},2.0\,\text{m}) = \frac{\sigma_2}{2\epsilon_0}\hat{r} = \frac{+45\,\text{nC/m}^2}{2(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{r} = (2.54\,\text{kN/C})\hat{r}$$

where $\hat{r}$ is a unit vector pointing from plane 2 toward the point whose coordinates are (6.0 m, 2.0 m).

Refer to the diagram below to obtain: $\hat{r} = \sin 30°\hat{i} - \cos 30°\hat{j}$




Substitute to obtain:

$$\begin{aligned}\vec{E}_2(6.0\,\text{m},2.0\,\text{m}) &= (2.54\,\text{kN/C})(\sin 30°\hat{i} - \cos 30°\hat{j}) \\ &= (1.27\,\text{kN/C})\hat{i} + (-2.20\,\text{kN/C})\hat{j}\end{aligned}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{E}(6.0\,\text{m},2.0\,\text{m}) &= (3.67\,\text{kN/C})\hat{j} + (1.27\,\text{kN/C})\hat{i} + (-2.20\,\text{kN/C})\hat{j} \\ &= \boxed{(1.3\,\text{kN/C})\hat{i} + (1.5\,\text{kN/C})\hat{j}}\end{aligned}$$

(*b*) Note that $\vec{E}_1(6.0\,\text{m},5.0\,\text{m}) = \vec{E}_1(6.0\,\text{m},2.0\,\text{m})$ so that:

$$\vec{E}_1(6\,\text{m},5\,\text{m}) = \frac{\sigma}{2\epsilon_0}\hat{j} = \frac{65\,\text{nC/m}^2}{2(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{j} = (3.67\,\text{kN/C})\hat{j}$$

Note also that $\vec{E}_2(6.0\,\text{m},5.0\,\text{m}) = -\vec{E}_2(6.0\,\text{m},2.0\,\text{m})$ so that:

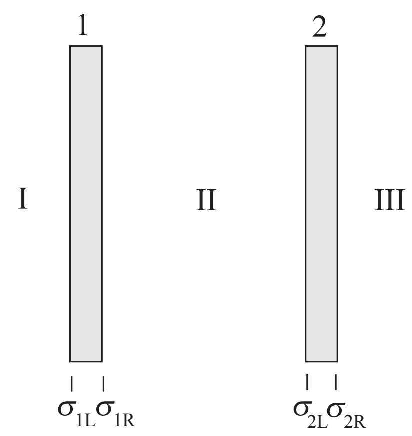
$$\vec{E}_2(6.0\,\text{m},5.0\,\text{m}) = (-1.27\,\text{kN/C})\hat{i} + (2.20\,\text{kN/C})\hat{j}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{E}(6.0\,\text{m},5.0\,\text{m}) &= (3.67\,\text{kN/C})\hat{j} + (-1.27\,\text{kN/C})\hat{i} + (2.20\,\text{kN/C})\hat{j} \\ &= \boxed{(-1.3\,\text{kN/C})\hat{i} + (5.9\,\text{kN/C})\hat{j}}\end{aligned}$$

**71 ••** Two identical square parallel metal plates each have an area of 500 cm$^2$. They are separated by 1.50 cm. They are both initially uncharged. Now a charge of $+1.50\,\text{nC}$ is transferred from the plate on the left to the plate on the right and the charges then establish electrostatic equilibrium. (Neglect edge effects.) (*a*) What is the electric field between the plates at a distance of 0.25 cm from the plate on the right? (*b*) What is the electric field between the plates a distance of 1.00 cm from the plate on the left? (*c*) What is the electric field just to the left of the plate on the left? (*d*) What is the electric field just to the right of the plate to the right?

**Picture the Problem** The transfer of charge from the plate on the left to the plate on the right leaves the plates with equal but opposite charges. Because the metal plates are conductors, the charge on each plate is completely on the surface facing the other plate. The symbols for the four surface charge densities are shown in the figure. The *x* component of the electric field due to the charge on surface 1L is $-\sigma_{1\text{L}}/2\epsilon_0$ at points to the left of surface 1L and is $+\sigma_{1\text{L}}/2\epsilon_0$ at points to the right of surface 1L, where the +*x* direction is to the right. Similar expressions describe the electric fields due to the other three surface charges. We can use superposition of electric fields to find the electric field in each of the three regions.




Define $\sigma_1$ and $\sigma_2$ so that:

$$\sigma_1 = \sigma_{1\text{L}} + \sigma_{1\text{R}}$$
and
$$\sigma_2 = \sigma_{2\text{L}} + \sigma_{2\text{R}}$$

(*a*) and (*b*) In the region between the plates (region II):

$$E_{x,\mathrm{II}} = \frac{\sigma_{1\mathrm{L}}}{2\epsilon_0} + \frac{\sigma_{1\mathrm{R}}}{2\epsilon_0} - \frac{\sigma_{2\mathrm{L}}}{2\epsilon_0} - \frac{\sigma_{2\mathrm{R}}}{2\epsilon_0}$$

$$= 0 + \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{2\epsilon_0} - 0 = \frac{\sigma_1 - \sigma_2}{\epsilon_0}$$

Let $\sigma_2 = -\sigma_1 = \sigma$. Then:

$$\sigma_1 - \sigma_2 = -\sigma - \sigma = -2\sigma$$

Substituting for $\sigma_1 - \sigma_2$ and using the definition of $\sigma$ yields:

$$E_{x,\mathrm{II}} = \frac{-2\sigma}{\epsilon_0} = -\frac{Q}{A\epsilon_0}$$

Substitute numerical values and evaluate $E_{x,\mathrm{II}}$:

$$E_{x,\mathrm{II}} = -\frac{1.50\,\mathrm{nC}}{\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}\right)\left(500\times10^{-6}\,\mathrm{m}^2\right)} = \boxed{339\,\mathrm{kN/C}} \text{ toward the left}$$

(*c*) The electric field strength just to the left of the plate on the left (region I) is given by:

$$E_{x,\mathrm{I}} = -\frac{\sigma_{1\mathrm{L}}}{2\epsilon_0} - \frac{\sigma_{1\mathrm{R}}}{2\epsilon_0} - \frac{\sigma_{2\mathrm{L}}}{2\epsilon_0} - \frac{\sigma_{2\mathrm{R}}}{2\epsilon_0}$$

$$= 0 - \frac{\sigma_1}{2\epsilon_0} - \frac{\sigma_2}{2\epsilon_0} - 0$$

$$= -\frac{-\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = \boxed{0}$$

(*d*) The electric field strength just to the right of the plate on the right (region III) is given by:

$$E_{x,\mathrm{III}} = \frac{\sigma_{1\mathrm{L}}}{2\epsilon_0} + \frac{\sigma_{1\mathrm{R}}}{2\epsilon_0} + \frac{\sigma_{2\mathrm{L}}}{2\epsilon_0} + \frac{\sigma_{2\mathrm{R}}}{2\epsilon_0}$$

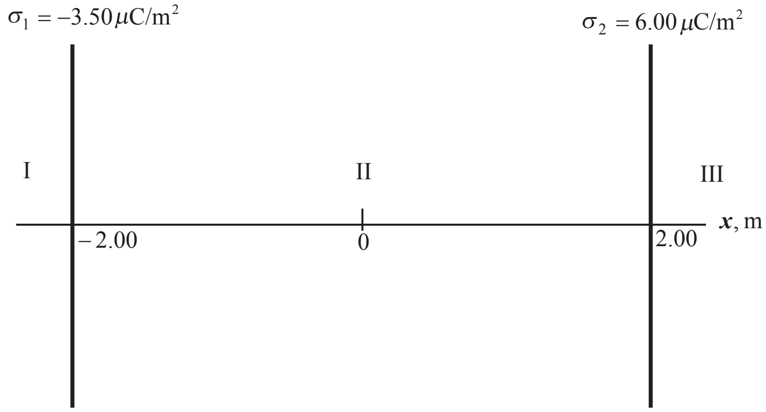
$$= 0 + \frac{\sigma_1}{2\epsilon_0} + \frac{\sigma_2}{2\epsilon_0} - 0$$

$$= \frac{-\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \boxed{0}$$

**72** •• Two infinite nonconducting uniformly charged planes lie parallel to each other and to the *yz* plane. One is at $x = -2.00$ m and has a surface charge density of $-3.50\ \mu\mathrm{C/m^2}$. The other is at $x = 2.00$ m and has a surface charge density of $6.00\ \mu\mathrm{C/m^2}$. Find the electric field in the region (*a*) $x < -2.00$ m, (*b*) $-2.00\text{ m} < x < 2.00$ m, and (*c*) $x > 2.00$ m.

**Picture the Problem** Let the numeral 1 refer to the infinite plane at $x = -2.00$ m and the numeral 2 to the plane at $x = 2.00$ m and let I, II, and III identify the regions to the left of plane 1, between the planes, and to the right of plane 2, respectively. We can use the expression for the electric field of in infinite plane of charge to express the electric field due to each plane of charge in each of the three

regions. Their sum will be the resultant electric field in each region.




The resultant electric field is the sum of the fields due to planes 1 and 2:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \quad (1)$$

(*a*) The field due to plane 1 in region I is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\in_0}\left(-\hat{i}\right)$$

The field due to plane 2 in region I is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\in_0}\left(-\hat{i}\right)$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\in_0}\left(-\hat{i}\right) + \frac{\sigma_2}{2\in_0}\left(-\hat{i}\right) = -\frac{\sigma_1 + \sigma_2}{2\in_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\,\mu\text{C/m}^2 + 6.00\,\mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = \boxed{\left(-141\,\text{kN/C}\right)\hat{i}}$$

(*b*) The field due to plane 1 in region II is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\in_0}\hat{i}$$

The field due to plane 2 in region II is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\in_0}\left(-\hat{i}\right)$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\in_0}\left(\hat{i}\right) + \frac{\sigma_2}{2\in_0}\left(-\hat{i}\right) = \frac{\sigma_1 - \sigma_2}{2\in_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\,\mu\text{C/m}^2 - 6.00\,\mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = \boxed{(-536\,\text{kN/C})\hat{i}}$$

(*c*) The field due to plane 1 in region III is given by:

$$\vec{E}_1 = \frac{\sigma_1}{2\in_0}\hat{i}$$

The field due to plane 2 in region III is given by:

$$\vec{E}_2 = \frac{\sigma_2}{2\in_0}\hat{i}$$

Substitute for $\vec{E}_1$ and $\vec{E}_2$ in equation (1) and simplify to obtain:

$$\vec{E} = \frac{\sigma_1}{2\in_0}\left(\hat{i}\right) + \frac{\sigma_2}{2\in_0}\left(\hat{i}\right) = \frac{\sigma_1+\sigma_2}{2\in_0}\hat{i}$$

Substitute numerical values and evaluate $\vec{E}$:

$$\vec{E} = -\frac{-3.50\,\mu\text{C/m}^2 + 6.00\,\mu\text{C/m}^2}{2\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{i} = \boxed{(141\,\text{kN/C})\hat{i}}$$

**73 ••• [SSM]** A quantum-mechanical treatment of the hydrogen atom shows that the electron in the atom can be treated as a smeared-out distribution of negative charge of the form $\rho(r) = -\rho_0 e^{-2r/a}$. Here *r* represents the distance from the center of the nucleus and *a* represents the first *Bohr radius* which has a numerical value of 0.0529 nm. Recall that the nucleus of a hydrogen atom consists of just one proton and treat this proton as a positive point charge. (*a*) Calculate $\rho_0$, using the fact that the atom is neutral. (*b*) Calculate the electric field at any distance *r* from the nucleus.

**Picture the Problem** Because the atom is uncharged, we know that the integral of the electron's charge distribution over all of space must equal its charge $q_e$. Evaluation of this integral will lead to an expression for $\rho_0$. In (*b*) we can express the resultant electric field at any point as the sum of the electric fields due to the proton and the electron cloud.

(*a*) Because the atom is uncharged, the integral of the electron's charge distribution over all of space must equal its charge *e*:

$$e = \int_0^\infty \rho(r)\,dV = \int_0^\infty \rho(r)4\pi r^2\,dr$$

Substitute for $\rho(r)$ and simplify to obtain:

$$e = -\int_0^\infty \rho_0 e^{-2r/a}\,4\pi r^2\,dr = -4\pi\rho_0\int_0^\infty r^2 e^{-2r/a}\,dr$$

Use integral tables or integration by parts to obtain:

$$\int_0^\infty r^2 e^{-2r/a}dr = \frac{a^3}{4}$$

Substitute for $\int_0^\infty r^2 e^{-2r/a}dr$ to obtain:

$$e = -4\pi\rho_0\left(\frac{a^3}{4}\right) = -\pi a^3\rho_0$$

Solving for $\rho_0$ yields:

$$\rho_0 = \boxed{-\frac{e}{\pi a^3}}$$

(*b*) The field will be the sum of the field due to the proton and that of the electron charge cloud:

$$E = E_{\text{p}} + E_{\text{cloud}}$$

Express the field due to the electron cloud:

$$E_{\text{cloud}}(r) = \frac{kQ(r)}{r^2}$$

where $Q(r)$ is the net negative charge enclosed a distance $r$ from the proton.

Substitute for $E_{\text{p}}$ and $E_{\text{cloud}}$ to obtain:

$$E(r) = \frac{ke}{r^2} + \frac{kQ(r)}{r^2} \qquad (1)$$

$Q(r)$ is given by:

$$Q(r) = \int_0^r 4\pi r'^2\rho(r')dr'$$

$$= 4\pi\int_0^r r'^2\rho_0 e^{-2r'/a}dr'$$

From Part (*a*), $\rho_0 = \dfrac{-e}{\pi a^3}$:

$$Q(r) = 4\pi\left(\frac{-e}{\pi a^3}\right)\int_0^r r'^2 e^{-2r'/a}dr'$$

$$= \frac{-4e}{a^3}\int_0^r r'^2 e^{-2r'/a}dr'$$

From a table of integrals:

$$\int_0^r x^2 e^{-2x/a}dx = \tfrac{1}{4}e^{-2r/a}a\left[\left(e^{-2r/a}-1\right)a^2 - 2ar - 2r^2\right]$$

$$= \tfrac{1}{4}e^{-2r/a}a^3\left[\left(e^{-2r/a}-1\right) - 2\frac{r}{a} - 2\frac{r^2}{a^2}\right]$$

$$= \frac{a^3}{4}\left[\left(1-e^{-2r/a}\right) - 2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$

Substituting for $\int_0^r r'^2 e^{-2r'/a}dr'$ in the expression for $Q(r)$ and simplifying yields:

$$Q(r)=\frac{-e}{4}\left[\left(1-e^{-2r/a}\right)-2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$

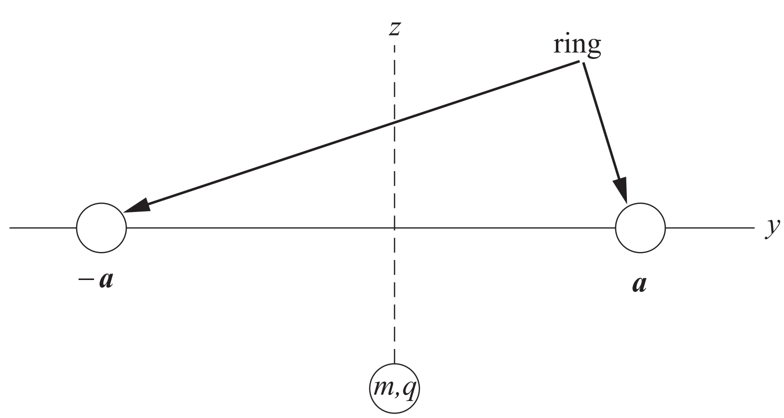
Substitute for $Q(r)$ in equation (1) and simplify to obtain:

$$E(r)=\frac{ke}{r^2}-\frac{ke}{4r^2}\left[\left(1-e^{-2r/a}\right)-2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]$$

$$=\boxed{\frac{ke}{r^2}\left(1-\frac{1}{4}\left[\left(1-e^{-2r/a}\right)-2e^{-2r/a}\left(\frac{r}{a}+\frac{r^2}{a^2}\right)\right]\right)}$$

**74** •• A uniformly charged ring has a radius *a,* lies in a horizontal plane, and has a negative charge given by –*Q*. A small particle of mass *m* has a positive charge given by *q*. The small particle is located on the axis of the ring. (*a*) What is the minimum value of *q/m* such that the particle will be in equilibrium under the action of gravity and the electrostatic force? (*b*) If *q/m* is twice the value calculated in Part (*a*), where will the particle be when it is in equilibrium? Express your answer in terms of *a*

**Picture the Problem** We can apply the condition for translational equilibrium to the particle and use the expression for the electric field on the axis of a ring charge to obtain an expression for *q/m*. Doing so will lead us to the conclusion that *q/m* will be a minimum when $E_z$ is a maximum and that $z=-a/\sqrt{2}$ maximizes $E_z$.



(*a*) Apply $\sum F_z = 0$ to the particle:

$$qE_z - mg = 0 \Rightarrow \frac{q}{m} = \frac{g}{E_z} \qquad (1)$$

Note that this result tells us that the minimum value of $q/m$ will be where the field due to the ring is greatest.

Express the electric field along the $z$ axis due to the ring of charge:

$$E_z = \frac{kQz}{\left(z^2 + a^2\right)^{3/2}}$$

Differentiate this expression with respect to $z$ to obtain:

$$\frac{dE_x}{dz} = kQ\frac{d}{dz}\left[\frac{z}{\left(z^2 + a^2\right)^{3/2}}\right] = kQ\frac{\left(z^2 + a^2\right)^{3/2} - z\frac{d}{dx}\left(z^2 + a^2\right)^{3/2}}{\left(z^2 + a^2\right)^3}$$

$$= kQ\frac{\left(z^2 + a^2\right)^{3/2} - z\left(\frac{3}{2}\right)\left(z^2 + a^2\right)^{1/2}(2z)}{\left(z^2 + a^2\right)^3} = kQ\frac{\left(z^2 + a^2\right)^{3/2} - 3z^2\left(z^2 + a^2\right)^{1/2}}{\left(z^2 + a^2\right)^3}$$

Set this expression equal to zero for extrema and simplify:

$$\frac{\left(z^2 + a^2\right)^{3/2} - 3z^2\left(z^2 + a^2\right)^{1/2}}{\left(z^2 + a^2\right)^3} = 0\,,$$

$$\left(z^2 + a^2\right)^{3/2} - 3z^2\left(z^2 + a^2\right)^{1/2} = 0\,,$$

and

$$z^2 + a^2 - 3z^2 = 0$$

Solve for $z$ to obtain:

$$z = \pm\frac{a}{\sqrt{2}}$$

as candidates for maxima or minima.

You can either plot a graph of $E_z$ or evaluate its second derivative at these points to show that it is a maximum at:

$$z = -\frac{a}{\sqrt{2}}$$

Substitute to obtain an expression $E_{z,\max}$:

$$E_{z,\max} = \left|\frac{kQ\left(-\frac{a}{\sqrt{2}}\right)}{\left(\left(-\frac{a}{\sqrt{2}}\right)^2 + a^2\right)^{3/2}}\right| = \frac{2kQ}{\sqrt{27}a^2}$$

Substitute in equation (1) to obtain:

$$\frac{q}{m} = \boxed{\frac{\sqrt{27}\,ga^2}{2kQ}}$$

($b$) If $q/m$ is twice as great as in ($a$), then the electric field should be half its value in ($a$), that is:

$$\frac{kQ}{\sqrt{27}a^2} = \frac{kQz}{\left(z^2+a^2\right)^{3/2}}$$

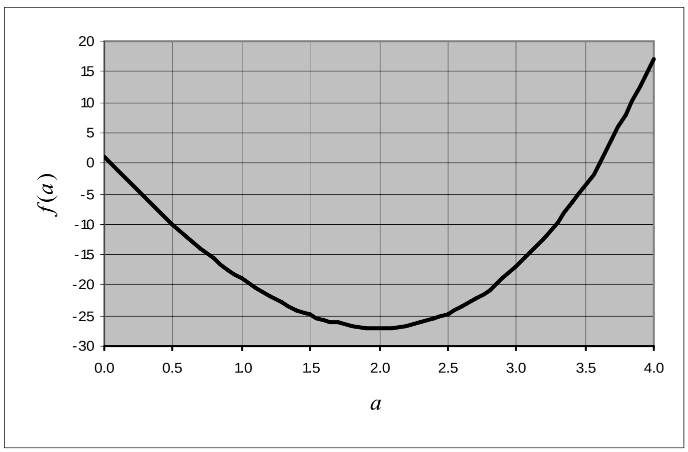
or

$$\frac{1}{27a^4} = \frac{z^2}{a^6\left(1+\frac{z^2}{a^2}\right)^3}$$

Let $u = z^2/a^2$ and simplify to obtain:

$$u^3 + 3u^2 - 24u + 1 = 0$$

The following graph of $f(u) = u^3 + 3u^2 - 24u + 1$ was plotted using a spreadsheet program.




Use your graphing calculator or trial-and-error methods to obtain:

$\boldsymbol{u} = 0.04189$ and $\boldsymbol{u} = 3.596$

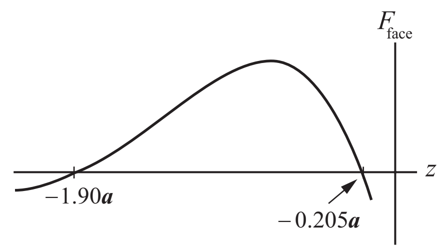
The corresponding $z$ values are:

$\boldsymbol{z} = -0.205\boldsymbol{a}$ and $\boldsymbol{z} = -1.90\boldsymbol{a}$

The condition for a stable equilibrium position is that the particle, when displaced from its equilibrium position, experiences a restoring force; that is, a force that acts toward the equilibrium position. When the particle in this problem is just above its equilibrium position the net force on it must be downward and when it is just below the equilibrium position the net force on it must be upward. Note that the electric force is zero at the origin, so the net force there is downward and remains downward to the first equilibrium position as the weight force exceeds the electric force in this interval. The net force is upward between the first and

second equilibrium positions as the electric force exceeds the weight force. The net force is downward below the second equilibrium position as the weight force exceeds the electric force. Thus, the first (higher) equilibrium position is stable and the second (lower) equilibrium position is unstable.

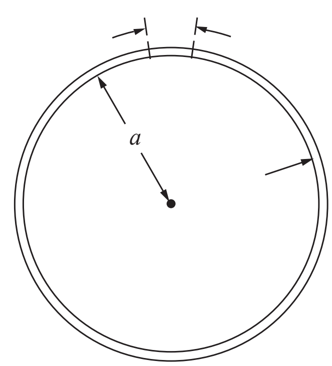
You might also find it instructive to use your graphing calculator to plot a graph of the electric force (the gravitational force is constant and only shifts the graph of the total force downward). Doing so will produce a graph similar to the one shown in the sketch to the right.




Note that the slope of the graph is negative on both sides of $-0.205a$ whereas it is positive on both sides of $-1.90a$; further evidence that $-0.205a$ is a position of stable equilibrium and $-1.90a$ a position of unstable equilibrium.

**75** •• A long, thin, non-conducting plastic rod is bent into a circular loop that has a radius $a$. Between the ends of the rod a short gap of length $\ell$, where $\ell << a$, remains. A positive charge of magnitude $Q$ is evenly distributed on the loop. ($a$) What is the direction of the electric field at the center of the loop? Explain your answer. ($b$) What is the magnitude of the electric field at the center of the loop?

**Picture the Problem** The loop with the small gap is equivalent to a closed loop and a charge of $-Q\ell/(2\pi R)$ at the gap. The field at the center of a closed loop of uniform line charge is zero. Thus the field is entirely due to the charge $-Q\ell/(2\pi R)$.




($a$) Express the field at the center of the loop:

$$\vec{E}_{\text{center}} = \vec{E}_{\text{loop}} + \vec{E}_{\text{gap}} \qquad (1)$$

Relate the field at the center of the loop to the charge in the gap:

$$\vec{E}_{\text{gap}} = -\frac{kq}{a^2}\hat{r}$$

Use the definition of linear charge

$$\lambda = \frac{q}{\ell} = \frac{Q}{2\pi a} \Rightarrow q = \frac{Q\ell}{2\pi a}$$

density to relate the charge in the gap to the length of the gap:

Substitute for $q$ to obtain:

$$\vec{E}_{\text{gap}} = -\frac{kQ\ell}{2\pi a^3}\hat{r}$$

Substituting in equation (1) yields:

$$\vec{E}_{\text{center}} = 0 - \frac{kQ\ell}{2\pi a^3}\hat{r} = -\frac{kQ\ell}{2\pi a^3}\hat{r}$$

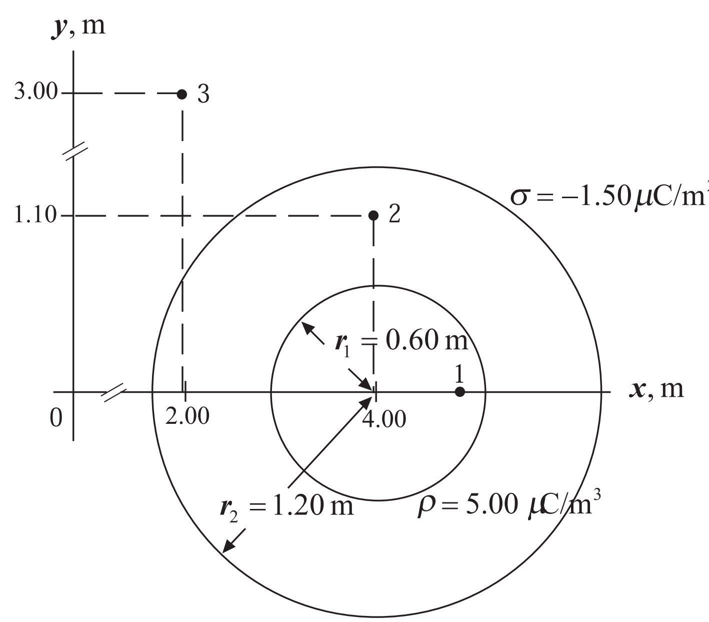
If $Q$ is positive, the field at the origin points radially outward toward the gap.

($b$) From our result in ($a$) we see that the magnitude of $\vec{E}_{\text{center}}$ is:

$$E_{\text{center}} = \boxed{\frac{kQ\ell}{2\pi a^3}}$$

**76** •• A non-conducting solid sphere that is 1.20 m in diameter and has its center on the $x$ axis at $x = 4.00$ m has a uniform volume charge of density of $+5.00\ \mu\text{C/m}^3$. Concentric with the sphere is a thin non-conducting spherical shell that has a diameter of 2.40 m and a uniform surface charge density of $-1.50\ \mu\text{C/m}^2$. Calculate the magnitude and direction of the electric field at ($a$) $x = 4.50$ m, $y = 0$, ($b$) $x = 4.00$ m, $y = 1.10$ m, and ($c$) $x = 2.00$ m, $y = 3.00$ m.

**Picture the Problem** We can find the electric fields at the three points of interest, labeled 1, 2, and 3 in the diagram, by adding the electric fields due to the charge distributions on the nonconducting sphere and the spherical shell.




Express the electric field due to the

$$\vec{E} = \vec{E}_{\text{sphere}} + \vec{E}_{\text{shell}} \qquad (1)$$

nonconducting sphere and the spherical shell at any point in space:

(*a*) Because (4.50 m, 0) is inside the spherical shell:

$$\vec{E}_{\text{shell}}(4.50\,\text{m},0) = 0$$

Apply Gauss's law to a spherical surface inside the nonconducting sphere to obtain:

$$\vec{E}_{\text{sphere}}(r) = \frac{4\pi}{3}k\rho r\hat{\boldsymbol{i}}$$

Evaluate $\vec{E}_{\text{sphere}}(0.50\,\text{m})$:

$$\vec{E}_{\text{sphere}}(0.50\,\text{m}) = \frac{4\pi}{3}\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(5.00\,\mu\text{C/m}^2\right)(0.50\,\text{m})\hat{\boldsymbol{i}} = (94.1\,\text{kN/C})\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\vec{E}(4.50\,\text{m},0) = (94.1\,\text{kN/C})\hat{\boldsymbol{i}} + 0 = (94.1\,\text{kN/C})\hat{\boldsymbol{i}}$$

Find the magnitude and direction of $\vec{E}(4.50\,\text{m},0)$:

$$E(4.50\,\text{m},0) = \boxed{94\,\text{kN/C}}$$

and

$$\theta = \boxed{0^\circ}$$

(*b*) Because (4.00 m, 1.10 m) is inside the spherical shell:

$$\vec{E}_{\text{shell}}(4.00\,\text{m},1.10\,\text{m}) = 0$$

Evaluate $\vec{E}_{\text{sphere}}(1.10\,\text{m})$:

$$\vec{E}_{\text{sphere}}(1.10\,\text{m}) = \frac{4\pi\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(5.00\,\mu\text{C/m}^2\right)(0.600\,\text{m})^3}{3(1.10\,\text{m})^2}\hat{\boldsymbol{j}} = (33.6\,\text{kN/C})\hat{\boldsymbol{j}}$$

Substitute in equation (1) to obtain:

$$\vec{E}(4.00\,\text{m},1.10\,\text{m}) = (33.6\,\text{kN/C})\hat{\boldsymbol{j}} + 0 = (33.6\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the magnitude and direction of $\vec{E}(4.00\,\text{m}, 1.10\,\text{m})$:

$E(4.00\,\text{m}, 1.10\,\text{m}) = \boxed{33.6\,\text{kN/C}}$

and

$\theta = \boxed{90°}$

($c$) Because (2.00 m, 3.00 m) outside the spherical shell:

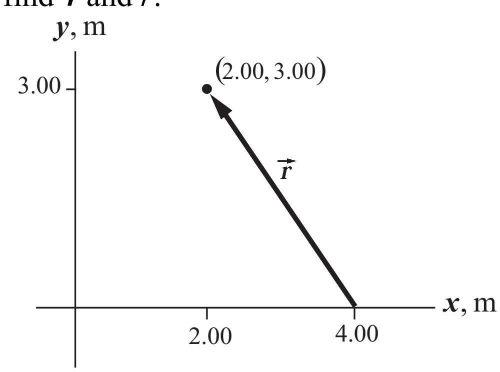
$$\vec{E}_{\text{shell}}(r) = \frac{kQ_{\text{shell}}}{r^2}\hat{r}$$

where $\hat{r}$ is a unit vector pointing from (4.00 m, 0) to (2.00 m, 3.00 m).

Evaluate $Q_{\text{shell}}$:

$$\begin{aligned} Q_{\text{shell}} &= \sigma A_{\text{shell}} \\ &= 4\pi(-1.50\,\mu\text{C/m}^2)(1.20\,\text{m})^2 \\ &= -27.14\,\mu\text{C} \end{aligned}$$

Refer to the following diagram to find $\hat{r}$ and $r$:




$r = 3.606\,\text{m}$

and

$\hat{r} = -0.5547\hat{i} + 0.8321\hat{j}$

Substitute numerical values and evaluate $\vec{E}_{\text{shell}}(2.00\,\text{m}, 3.00\,\text{m})$:

$$\begin{aligned} \vec{E}_{\text{shell}}(3.606\,\text{m}) &= \frac{(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(-27.14\,\mu\text{C})}{(3.606\,\text{m})^2}\hat{r} \\ &= (-18.77\,\text{kN/C})(-0.5547\hat{i} + 0.8321\hat{j}) \\ &= (10.41\,\text{kN/C})\hat{i} + (-15.61\,\text{kN/C})\hat{j} \end{aligned}$$

Express the electric field due to the charged nonconducting sphere at a distance $r$ from its center that is greater than its radius:

$$\vec{E}_{\text{sphere}}(r) = \frac{kQ_{\text{sphere}}}{r^2}\hat{r}$$

Find the charge on the sphere:

$$\begin{aligned} Q_{\text{sphere}} &= \rho V_{\text{sphere}} \\ &= \frac{4\pi}{3}\left(5.00\ \mu\text{C/m}^2\right)\left(0.600\ \text{m}\right)^3 \\ &= 4.524\ \mu\text{C} \end{aligned}$$

Evaluate $\vec{E}_{\text{sphere}}(3.61\ \text{m})$:

$$\begin{aligned} \vec{E}_{\text{sphere}}(2.00\ \text{m},3.00\ \text{m}) &= \frac{\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(4.524\ \mu\text{C}\right)}{\left(3.606\ \text{m}\right)^2}\hat{\boldsymbol{r}} = \left(3.128\ \text{kN/C}\right)\hat{\boldsymbol{r}} \\ &= \left(3.128\ \text{kN/C}\right)\left(-0.5547\hat{\boldsymbol{i}} + 0.8321\hat{\boldsymbol{j}}\right) \\ &= \left(-1.735\ \text{kN/C}\right)\hat{\boldsymbol{i}} + \left(2.602\ \text{kN/C}\right)\hat{\boldsymbol{j}} \end{aligned}$$

Substitute in equation (1) to obtain:

$$\begin{aligned} \vec{E}(2.00\ \text{m},3.00\ \text{m}) &= \left(10.41\ \text{kN/C}\right)\hat{\boldsymbol{i}} + \left(-15.61\ \text{kN/C}\right)\hat{\boldsymbol{j}} + \left(-1.735\ \text{kN/C}\right)\hat{\boldsymbol{i}} \\ &\quad + \left(2.602\ \text{kN/C}\right)\hat{\boldsymbol{j}} \\ &= \left(8.675\ \text{kN/C}\right)\hat{\boldsymbol{i}} + \left(-13.01\ \text{kN/C}\right)\hat{\boldsymbol{j}} \end{aligned}$$

Find the magnitude and direction of $\vec{E}(2.00\ \text{m},3.00\ \text{m})$:

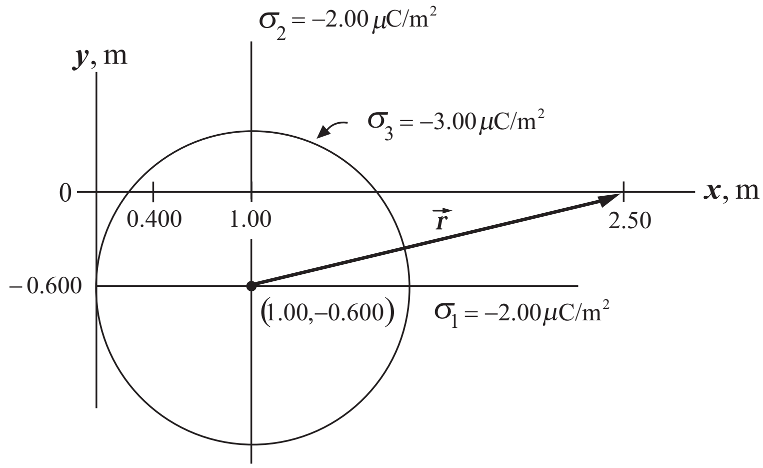
$$E(2.00\ \text{m},3.00\ \text{m}) = \sqrt{\left(8.675\ \text{kN/C}\right)^2 + \left(-13.01\ \text{kN/C}\right)^2} = \boxed{15.6\ \text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{-13.01\ \text{kN/C}}{8.675\ \text{kN/C}}\right) = \boxed{304°}$$

**77 ••** An infinite non-conducting plane sheet of charge that has a surface charge density + 3.00 $\mu$C/m$^2$ lies in the $y$ = –0.600 m plane. A second infinite non-conducting plane sheet of charge that has a surface charge density of –2.00 $\mu$C/m$^2$ lies in the $x$ = 1.00 m plane. Lastly, a non-conducting thin spherical shell of radius of 1.00 m and that has its center in the $z$ = 0 plane at the intersection of the two charged planes has a surface charge density of –3.00 $\mu$C/m$^2$. Find the magnitude and direction of the electric field on the $x$ axis at ($a$) $x$ = 0.400 m and ($b$) $x$ = 2.50 m.

**Picture the Problem** Let the numeral 1 refer to the infinite plane whose charge density is $\sigma_1$ and the numeral 2 to the infinite plane whose charge density is $\sigma_2$. We can find the electric fields at the two points of interest by adding the electric fields due to the charge distributions on the infinite planes and the sphere.




Express the electric field due to the infinite planes and the sphere at any point in space:

$$\vec{E} = \vec{E}_{\text{sphere}} + \vec{E}_1 + \vec{E}_2 \qquad (1)$$

(*a*) Because (0.400 m, 0) is inside the sphere:

$$\vec{E}_{\text{sphere}}(0.400\,\text{m},0) = 0$$

Find the field at (0.400 m, 0) due to plane 1:

$$\vec{E}_1(0.400\,\text{m},0) = \frac{\sigma_1}{2\epsilon_0}\hat{j} = \frac{3.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{j} = (169.4\,\text{kN/C})\hat{j}$$

Find the field at (0.400 m, 0) due to plane 2:

$$\vec{E}_2(0.400\,\text{m},0) = \frac{\sigma_2}{2\epsilon_0}(-\hat{i}) = \frac{-2.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}(-\hat{i}) = (112.9\,\text{kN/C})\hat{i}$$

Substitute in equation (1) to obtain:

$$\vec{E}(0.400\,\text{m},0) = 0 + (169.4\,\text{kN/C})\hat{j} + (112.9\,\text{kN/C})\hat{i} = (112.9\,\text{kN/C})\hat{i} + (169.4\,\text{kN/C})\hat{j}$$

Find the magnitude and direction of $\vec{E}(0.400\,\text{m},0)$:

$$E(0.400\,\text{m},0) = \sqrt{(112.9\,\text{kN/C})^2 + (169.4\,\text{kN/C})^2} = 203.6\,\text{kN/C} = \boxed{204\,\text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{169.4\,\text{kN/C}}{112.9\,\text{kN/C}}\right) = 56.31^\circ = \boxed{56.3^\circ}$$

(*b*) Because (2.50 m, 0) is outside the sphere:

$$\vec{\boldsymbol{E}}_{\text{sphere}}(0.400\,\text{m},0)=\frac{\boldsymbol{k}\boldsymbol{Q}_{\text{sphere}}}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing from (1.00 m, –0.600 m) to (2.50 m, 0).

Evaluate $Q_{\text{sphere}}$:

$$\begin{aligned}Q_{\text{sphere}}&=\sigma A_{\text{sphere}}=4\pi\sigma R^2\\&=4\pi\left(-3.00\,\mu\text{C/m}^2\right)(1.00\,\text{m})^2\\&=-37.70\,\mu\text{C}\end{aligned}$$

Referring to the diagram above, determine $r$ and $\hat{\boldsymbol{r}}$:

$\boldsymbol{r}=1.616\,\text{m}$

and

$\hat{\boldsymbol{r}}=0.9285\hat{\boldsymbol{i}}+0.3714\hat{\boldsymbol{j}}$

Substitute and evaluate $\vec{\boldsymbol{E}}_{\text{sphere}}(2.50\,\text{m},0)$:

$$\begin{aligned}\vec{\boldsymbol{E}}_{\text{sphere}}(2.50\,\text{m},0)&=\frac{\left(8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(-37.70\,\boldsymbol{\mu C}\right)}{(1.616\,\text{m})^2}\hat{\boldsymbol{r}}\\&=(-129.8\,\text{kN/C})\left(0.9285\hat{\boldsymbol{i}}+0.3714\hat{\boldsymbol{j}}\right)\\&=(-120.5\,\text{kN/C})\hat{\boldsymbol{i}}+(-48.22\,\text{kN/C})\hat{\boldsymbol{j}}\end{aligned}$$

Find the field at (2.50 m, 0) due to plane 1:

$$\vec{\boldsymbol{E}}_1(2.50\,\text{m},0)=\frac{\boldsymbol{\sigma}_1}{2\in_0}\hat{\boldsymbol{j}}=\frac{3.00\,\boldsymbol{\mu}\text{C/m}^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{\boldsymbol{j}}=(169.4\,\text{kN/C})\hat{\boldsymbol{j}}$$

Find the field at (2.50 m, 0) due to plane 2:

$$\vec{\boldsymbol{E}}_2(2.50\,\text{m},0)=\frac{\boldsymbol{\sigma}_2}{2\in_0}\hat{\boldsymbol{i}}=\frac{-2.00\,\boldsymbol{\mu}\text{C/m}^2}{2\left(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}\hat{\boldsymbol{i}}=(-112.9\,\text{kN/C})\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\begin{aligned}\vec{\boldsymbol{E}}(0.400\,\text{m},0)&=(-120.5\,\text{kN/C})\hat{\boldsymbol{i}}+(-48.19\,\text{kN/C})\hat{\boldsymbol{j}}+(169.4\,\text{kN/C})\hat{\boldsymbol{j}}+(-112.9\,\text{kN/C})\hat{\boldsymbol{i}}\\&=(-233.5\,\text{kN/C})\hat{\boldsymbol{i}}+(121.2\,\text{kN/C})\hat{\boldsymbol{j}}\end{aligned}$$

Find the magnitude and direction of $\vec{E}(2.50\,\text{m},0)$:

$$E(2.50\,\text{m},0)=\sqrt{(-233.5\,\text{kN/C})^2+(121.2\,\text{kN/C})^2}=\boxed{263\,\text{kN/C}}$$

and

$$\theta=\tan^{-1}\left(\frac{121.2\,\text{kN/C}}{-233.5\,\text{kN/C}}\right)=\boxed{153°}$$

**78** •• An infinite non-conducting plane sheet lies in the $x$ = 2.00 m plane and has a uniform surface charge density of +2.00 $\mu$C/m$^2$. An infinite non-conducting line charge of uniform linear charge density 4.00 $\mu$C/m passes through the origin at an angle of 45.0° with the $x$ axis in the $xy$ plane. A solid non-conducting sphere of volume charge density –6.00 $\mu$C/m$^3$ and radius 0.800 m is centered on the $x$ axis at $x$ = 1.00 m. Calculate the magnitude and direction of the electric field in the $z$ = 0 plane at $x$ = 1.50 m, $y$ = 0.50 m.

**Picture the Problem** Let $P$ represent the point of interest at (1.50 m, 0.50 m). We can find the electric field at $P$ by adding the electric fields due to the infinite plane, the infinite line, and the sphere. Once we've expressed the field at $P$ in vector form, we can find its magnitude and direction.

Express the electric field at $P$:

$$\vec{E}=\vec{E}_{\text{plane}}+\vec{E}_{\text{line}}+\vec{E}_{\text{sphere}}$$

Find $\vec{E}_{\text{plane}}$ at $P$:

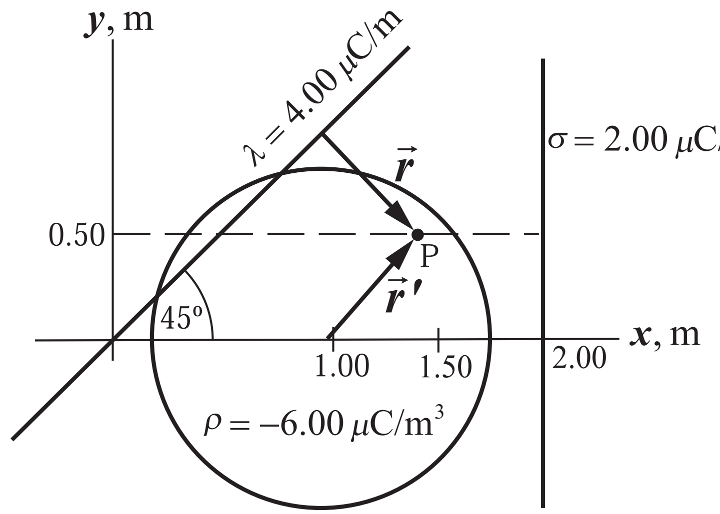
$$\vec{E}_{\text{plane}}=-\frac{\sigma}{2\epsilon_0}\hat{i}=-\frac{2.00\,\mu\text{C/m}^2}{2(8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)}\hat{i}=(-112.9\,\text{kN/C})\hat{i}$$

Express $\vec{E}_{\text{line}}$ at $P$:

$$\vec{E}_{\text{line}}=\frac{2k\lambda}{r}\hat{r}$$

Refer to the following figure to obtain:

$$\vec{r}=(0.50\,\text{m})\hat{i}-(0.50\,\text{m})\hat{j}\ \text{and}\ \hat{r}=(0.707)\hat{i}-(0.707)\hat{j}$$




Substitute and simplify to obtain:

$$\vec{E}_{\text{line}} = \frac{2\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(4.00\ \mu\text{C/m}\right)}{0.707\,\text{m}}\left[(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}\right]$$
$$= (101.7\,\text{kN/C})\left[(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}\right] = (71.90\,\text{kN/C})\hat{\boldsymbol{i}} + (-71.90\,\text{kN/C})\hat{\boldsymbol{j}}$$

Letting $r'$ represent the distance from the center of the sphere to $P$, apply Gauss's law to a spherical surface of radius $r'$ centered at (1 m, 0) to obtain an expression for $\vec{E}_{\text{sphere}}$ at $P$:

$$\vec{E}_{\text{sphere}} = \frac{4\pi}{3}kr'\rho\hat{\boldsymbol{r}}'$$

where $\hat{\boldsymbol{r}}'$ is directed toward the center of the sphere.

Refer to the diagram used above to obtain:

$$\vec{\boldsymbol{r}}' = -(0.50\,\text{m})\hat{\boldsymbol{i}} - (0.50\,\text{m})\hat{\boldsymbol{j}}$$

and

$$\hat{\boldsymbol{r}}' = -(0.707)\hat{\boldsymbol{i}} - (0.707)\hat{\boldsymbol{j}}$$

Substitute numerical values and simplify to obtain:

$$\vec{E}_{\text{sphere}} = \frac{4\pi}{3}\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)(0.707\,\text{m})\left(-6.00\ \mu\text{C/m}^3\right)\left[(0.707)\hat{\boldsymbol{i}} + (0.707)\hat{\boldsymbol{j}}\right]$$
$$= (-112.9\,\text{kN/C})\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) = (-112.9\,\text{kN/C})\hat{\boldsymbol{i}} + (-112.9\,\text{kN/C})\hat{\boldsymbol{j}}$$

Evaluating $\vec{E}$ yields:

$$\vec{E} = (-112.9\,\text{kN/C})\hat{\boldsymbol{i}} + (71.90\,\text{kN/C})\hat{\boldsymbol{i}} + (-71.90\,\text{kN/C})\hat{\boldsymbol{j}} + (-112.9\,\text{kN/C})\hat{\boldsymbol{i}}$$
$$+ (-112.9\,\text{kN/C})\hat{\boldsymbol{j}}$$
$$= (-154.0\,\text{kN/C})\hat{\boldsymbol{i}} + (-184.9\,\text{kN/C})\hat{\boldsymbol{j}}$$

Finally, find the magnitude and direction of $\vec{E}$:

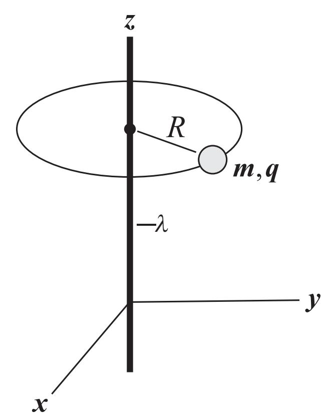
$$E = \sqrt{(-154.0\,\text{kN/C})^2 + (-184.9\,\text{kN/C})^2} = \boxed{241\,\text{kN/C}}$$

and

$$\theta = \tan^{-1}\left(\frac{-154.0\,\text{kN/C}}{-184.9\,\text{kN/C}}\right) = \boxed{220°}$$

**79 •• [SSM]** A uniformly charged, infinitely long line of negative charge has a linear charge density of $-\lambda$ and is located on the $z$ axis. A small positively charged particle that has a mass $m$ and a charge $q$ is in a circular orbit of radius $R$ in the $xy$ plane centered on the line of charge. ($a$) Derive an expression for the speed of the particle. ($b$) Obtain an expression for the period of the particle's orbit.

**Picture the Problem** ($a$) We can apply Newton's 2nd law to the particle to express its speed as a function of its mass $m$, charge $q$, and the radius of its path $R$, and the strength of the electric field due to the infinite line charge $E$. ($b$) The period of the particle's motion is the ratio of the circumference of the circle in which it travels divided by its orbital speed.




($a$) Apply Newton's 2nd law to the particle to obtain:

$$\sum F_{\text{radial}} = qE = m\frac{v^2}{R}$$

where the inward direction is positive.

Solving for $v$ yields:

$$v = \sqrt{\frac{qRE}{m}}$$

The strength of the electric field at a distance $R$ from the infinite line charge is given by:

$$E = \frac{2k\lambda}{R}$$

Substitute for $E$ and simplify to obtain:

$$v = \boxed{\sqrt{\frac{2kq\lambda}{m}}}$$

(*b*) The speed of the particle is equal to the circumference of its orbit divided by its period:

$$v=\frac{2\pi R}{T}\Rightarrow T=\frac{2\pi R}{v}$$

Substitute for $v$ and simplify to obtain:

$$T=\boxed{\pi R\sqrt{\frac{2m}{kq\lambda}}}$$

**80** •• A stationary ring of radius $a$ that lies in the $yz$ plane has a uniformly distributed positive charge $Q$. A small particle that has mass $m$ and a negative charge $q$ is located at the center of the ring. (*a*) Show that if $x \ll a$, the electric field along the axis of the ring is proportional to $x$. (*b*) Find the force on the particle of mass $m$ as a function of $x$. (*c*) Show that if the particle is given a small displacement in the $+x$ direction, it will perform simple harmonic motion. (*d*) What is the frequency of that motion?

**Picture the Problem** Starting with the equation for the electric field on the axis of ring charge, we can factor the denominator of the expression to show that, for $x \ll a$, $E_x$ is proportional to $x$. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's 2nd law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the period of the motion from its angular frequency, which we can obtain from the differential equation of motion.

(*a*) Express the electric field on the axis of the ring of charge:

$$E_x=\frac{kQx}{\left(x^2+a^2\right)^{3/2}}$$

Factor $a^2$ from the denominator of $E_x$ to obtain:

$$E_x=\frac{kQx}{\left[a^2\left(1+\frac{x^2}{a^2}\right)\right]^{3/2}}=\frac{kQx}{a^3\left(1+\frac{x^2}{a^2}\right)^{3/2}}\approx\boxed{\frac{kQ}{a^3}x}$$

provided $x \ll a$.

(*b*) Express the force acting on the particle as a function of its charge and the electric field:

$$F_x=qE_x=\boxed{\frac{kqQ}{a^3}x}$$

(*c*) Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's 2nd law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2} = -\frac{kqQ}{a^3}x$$

or

$$\boxed{\frac{d^2x}{dt^2} + \frac{kqQ}{ma^3}x = 0}$$

the differential equation of simple harmonic motion.

(*d*) Relate the frequency of the simple harmonic motion to its angular frequency:

$$f = \frac{\omega}{2\pi} \qquad (1)$$

From the differential equation we have:

$$\omega^2 = \frac{kqQ}{ma^3} \Rightarrow \omega = \sqrt{\frac{kqQ}{ma^3}}$$

Substitute for $\omega$ in equation (1) and simplify to obtain:

$$f = \boxed{\frac{1}{2\pi}\sqrt{\frac{kqQ}{ma^3}}}$$

**81 •• [SSM]** The charges $Q$ and $q$ of Problem 80 are +5.00 $\mu$C and –5.00 $\mu$C, respectively, and the radius of the ring is 8.00 cm. When the particle is given a small displacement in the $x$ direction, it oscillates about its equilibrium position at a frequency of 3.34 Hz. (*a*) What is the particle's mass? (*b*) What is the frequency if the radius of the ring is doubled to 16.0 cm and all other parameters remain unchanged?

**Picture the Problem** Starting with the equation for the electric field on the axis of a ring charge, we can factor the denominator of the expression to show that, for $x << a$, $E_x$ is proportional to $x$. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's 2nd law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the angular frequency of the motion from the differential equation and use this expression to find the frequency of the motion when the radius of the ring is doubled and all other parameters remain unchanged.

(*a*) Express the electric field on the axis of the ring of charge:

$$E_x = \frac{kQx}{\left(x^2 + a^2\right)^{3/2}}$$

Factor $a^2$ from the denominator of $E_x$ to obtain:

$$E_x = \frac{kQx}{\left[a^2\left(1+\frac{x^2}{a^2}\right)\right]^{3/2}} = \frac{kQx}{a^3\left(1+\frac{x^2}{a^2}\right)^{3/2}} \approx \frac{kQ}{a^3}x$$

provided $x << a$.

Express the force acting on the particle as a function of its charge and the electric field:

$$F_x = qE_x = \frac{kqQ}{a^3}x$$

Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's 2nd law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2} = -\frac{kqQ}{a^3}x$$

or

$$\frac{d^2x}{dt^2} + \frac{kqQ}{ma^3}x = 0$$

the differential equation of simple harmonic motion.

The angular frequency of the simple harmonic motion of the particle is given by:

$$\omega = \sqrt{\frac{kqQ}{ma^3}} \tag{1}$$

Solving for $m$ yields:

$$m = \frac{kqQ}{\omega^2 a^3} = \frac{kqQ}{4\pi^2 f^2 a^3}$$

Substitute numerical values and evaluate $m$:

$$m = \frac{\left(8.988\times10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left|\left(-5.00\ \mu\text{C}\right)\right|\left(5.00\ \mu\text{C}\right)}{4\pi^2\left(3.34\ \text{s}^{-1}\right)^2\left(8.00\ \text{cm}\right)^3} = \boxed{0.997\ \text{kg}}$$

(*b*) Express the angular frequency of the motion if the radius of the ring is doubled:

$$\omega' = \sqrt{\frac{kqQ}{m(2a)^3}} \tag{2}$$

Divide equation (2) by equation (1) to obtain:

$$\frac{\omega'}{\omega}=\frac{2\pi f'}{2\pi f}=\frac{\sqrt{\dfrac{kqQ}{m(2a)^3}}}{\sqrt{\dfrac{kqQ}{ma^3}}}=\frac{1}{\sqrt{8}}$$

Solve for $f'$ to obtain:

$$f'=\frac{f}{\sqrt{8}}=\frac{3.34\text{ Hz}}{\sqrt{8}}=\boxed{1.18\text{Hz}}$$

**82** •• If the radius of the ring in Problem 80 is doubled while keeping the linear charge density on the ring the same, does the frequency of oscillation of the particle change? If so, by what factor does it change?

**Picture the Problem** Starting with the equation for the electric field on the axis of a ring charge, we can factor the denominator of the expression to show that, for $x << a$, $E_x$ is proportional to $x$. We can use $F_x = qE_x$ to express the force acting on the particle and apply Newton's 2nd law to show that, for small displacements from equilibrium, the particle will execute simple harmonic motion. Finally, we can find the angular frequency of the motion from the differential equation and use this expression to find its value when the radius of the ring is doubled while keeping the linear charge density on the ring constant.

Express the electric field on the axis of the ring of charge:

$$E_x=\frac{kQx}{\left(x^2+a^2\right)^{3/2}}$$

Factor $a^2$ from the denominator of $E_x$ to obtain:

$$E_x=\frac{kQx}{\left[a^2\left(1+\dfrac{x^2}{a^2}\right)\right]^{3/2}}$$

$$=\frac{kQx}{a^3\left(1+\dfrac{x^2}{a^2}\right)^{3/2}}\approx\frac{kQ}{a^3}x$$

provided $x << a$.

Express the force acting on the particle as a function of its charge and the electric field:

$$F_x=qE_x=\frac{kqQ}{a^3}x$$

Because the negatively charged particle experiences a linear restoring force, we know that its motion will be simple harmonic. Apply Newton's 2nd law to the negatively charged particle to obtain:

$$m\frac{d^2x}{dt^2}=-\frac{kqQ}{a^3}x$$

or

$$\frac{d^2x}{dt^2}+\frac{kqQ}{ma^3}x=0,$$

the differential equation of simple harmonic motion.

The angular frequency of the simple harmonic motion of the particle is given by:

$$\omega=\sqrt{\frac{kqQ}{ma^3}} \qquad (1)$$

Express the angular frequency of the motion if the radius of the ring is doubled while keeping the linear charge density constant (that is, doubling $Q$):

$$\omega'=\sqrt{\frac{kq(2Q)}{m(2a)^3}} \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{\omega'}{\omega}=\frac{2\pi f'}{2\pi f}=\frac{\sqrt{\dfrac{kq(2Q)}{m(2a)^3}}}{\sqrt{\dfrac{kqQ}{ma^3}}}=\frac{1}{2}$$

Yes. The frequency changes by a factor of 0.5.

**83** ••• A uniformly charged non-conducting solid sphere of radius $R$ has its center at the origin and has volume charge density of $\rho$. (*a*) Show that at a point within the sphere a distance $r$ from the center $E=\frac{\rho}{3\in_0}r\hat{r}$. (*b*) Material is removed from the sphere leaving a spherical cavity of radius $b = R/2$ with its center at $x = b$ on the $x$ axis (Figure 22-46). Calculate the electric field at points 1 and 2 shown in Figure 22-46. *Hint: Model the sphere-with-cavity as two uniform spheres of equal positive and negative charge densities.*

**Picture the Problem** In Part (*a*), you can apply Gauss's law to express $\vec{E}$ as a function of $r$ for the uniformly charged nonconducting sphere with its center at the origin. In Part (*b*), you can use the hint to express the field at a generic point $P(x,y)$ in the cavity as the sum of the fields due to equal positive and negative charge densities and then evaluate this expression at points 1 and 2.

(*a*) The electric field at a distance *r* from the center of the uniformly charged nonconducting sphere is given by:

$$\vec{E}_\rho = E\hat{\boldsymbol{r}} \qquad (1)$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing radially outward.

Apply Gauss's law to a spherical surface of radius *r* centered at the origin to obtain:

$$\oint_S E_n\,dA = E_\rho\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Relate $Q_{\text{inside}}$ to the charge density $\rho$:

$$\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r^3} \Rightarrow Q_{\text{inside}} = \tfrac{4}{3}\rho\pi r^3$$

Substitute for $Q_{\text{inside}}$:

$$E_\rho\left(4\pi r^2\right) = \frac{4\rho\pi r^3}{3\,\epsilon_0}$$

Solve for $E_\rho$ to obtain:

$$E_\rho = \frac{\rho r}{3\,\epsilon_0}$$

Substitute for *E* in equation (1) to obtain:

$$\vec{E}_\rho = \boxed{\frac{\rho}{3\,\epsilon_0} r\,\hat{\boldsymbol{r}}}$$

(*b*) The electric field at point *P*(*x*,*y*) is the sum of the electric fields due to the two charge distributions:

$$\vec{E} = \vec{E}_\rho + \vec{E}_{-\rho} = E_\rho\hat{\boldsymbol{r}} + E_{-\rho}\hat{\boldsymbol{r}}' \qquad (2)$$

where $\hat{\boldsymbol{r}}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

Apply Gauss's law to a spherical surface of radius $r'$ centered at $x = b = R/2$ to obtain:

$$\oint_S E_n\,dA = E_{-\rho}\left(4\pi r'^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Relate $Q_{\text{inside}}$ to the charge density $-\rho$:

$$-\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r'^3} \Rightarrow Q_{\text{inside}} = -\tfrac{4}{3}\rho\pi r'^3$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E_{-\rho}\left(4\pi r'^2\right) = -\frac{4\pi r'^3\rho}{3\,\epsilon_0}$$

Solving for $E_{-\rho}$ yields:

$$E_{-\rho} = -\frac{\rho}{3\,\epsilon_0} r'$$

Substitute for $E_\rho$ and $E_{-\rho}$ in equation (2) to obtain:

$$\vec{E} = \frac{\rho}{3\,\epsilon_0} r\,\hat{\boldsymbol{r}} - \frac{\rho}{3\epsilon_0} r'\,\hat{\boldsymbol{r}} \qquad (3)$$

The vectors $\vec{r} = r\hat{r}$ and $\vec{r}' = r'\hat{r}'$ are given by:

$\vec{r} = x\hat{i} + y\hat{j}$ and $\vec{r}' = (x-b)\hat{i} + y\hat{j}$
where $x$ and $y$ are the coordinates of any point in the cavity.

Substitute for $r\hat{r}$ and $r'\hat{r}'$ in equation (3) and simplify to obtain:

$$\vec{E} = \frac{\rho}{3\epsilon_0}\left(x\hat{i} + y\hat{j}\right) - \frac{\rho}{3\epsilon_0}\left[(x-b)\hat{i} + y\hat{j}\right] = \frac{\rho b}{3\epsilon_0}\hat{i}$$

Because $\vec{E}$ is independent of $x$ and $y$:

$$\vec{E}_1 = \vec{E}_2 = \boxed{\frac{\rho b}{3\epsilon_0}\hat{i}}$$

**84** ••• Show that the electric field throughout the cavity of Problem 83*b* is uniform and is given by $\vec{E} = \dfrac{\rho}{3\epsilon_0} b\hat{i}$.

**Picture the Problem** The electric field in the cavity is the sum of the electric field due to the uniform and positive charge distribution of the sphere whose radius is *a* and the electric field due to any charge in the spherical cavity whose radius is *b*. You can use the hint given in Problem 83 to express the field at a generic point *P*(*x,y*) in the cavity as the sum of the fields due to equal positive and negative charge densities to show that $\vec{E} = \dfrac{\rho}{3\epsilon_0} b\hat{i}$.

The electric field at point *P*(*x,y*) is the sum of the electric fields due to the two charge distributions:

$$\vec{E} = \vec{E}_\rho + \vec{E}_{-\rho} = E_\rho\hat{r} + E_{-\rho}\hat{r}' \qquad (1)$$

where $\hat{r}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

Apply Gauss's law to a spherical surface of radius $r'$ centered at $x = b = R/2$ to obtain:

$$\oint_S E_n dA = E_{-\rho}\left(4\pi r'^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0}$$

Relate $Q_{\text{inside}}$ to the charge density $-\rho$:

$$-\rho = \frac{Q_{\text{inside}}}{\frac{4}{3}\pi r'^3} \Rightarrow Q_{\text{inside}} = -\tfrac{4}{3}\rho\pi r'^3$$

Substitute for $Q_{\text{inside}}$ to obtain:

$$E_{-\rho}\left(4\pi r'^2\right) = -\frac{4\pi r'^3\rho}{3\epsilon_0}$$

Solving for $E_{-\rho}$ yields:

$$E_{-\rho} = -\frac{\rho}{3\epsilon_0} r'$$

From Problem 83:

$$E_\rho = \frac{\rho}{3\in_0} r$$

Substitute for $E_\rho$ and $E_{-\rho}$ in equation (1) to obtain:

$$\vec{\boldsymbol{E}} = \frac{\rho}{3\in_0} r\hat{\boldsymbol{r}} - \frac{\rho}{3\in_0} r'\hat{\boldsymbol{r}} \qquad (2)$$

The vectors $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}' = r'\hat{\boldsymbol{r}}'$ are given by:

$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and $\vec{\boldsymbol{r}}' = (x-b)\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$
where $x$ and $y$ are the coordinates of any point in the cavity.

Substitute for $r\hat{\boldsymbol{r}}$ and $r'\hat{\boldsymbol{r}}'$ in equation (2) and simplify to obtain:

$$\vec{\boldsymbol{E}} = \frac{\rho}{3\in_0}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right) - \frac{\rho}{3\in_0}\left[(x-b)\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}\right] = \boxed{\frac{\rho}{3\in_0} b\hat{\boldsymbol{i}}}$$

**85** ••• The cavity in Problem 83*b* is now filled with a uniformly charged non-conducting material with a total charge of $Q$. Calculate the new values of the electric field at points 1 and 2 shown in Figure 22-46.

**Picture the Problem** The electric field at a generic point $P(x,y)$ in the cavity is the sum of the fields due to the positive charge density and the total charge $Q$.

The electric field at point $P(x,y)$ is the sum of the electric fields due to the two charge distributions:

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_\rho + \vec{\boldsymbol{E}}_{-\rho} + \vec{\boldsymbol{E}}_Q \qquad (1)$$

where $\hat{\boldsymbol{r}}'$ is a unit vector normal to a spherical Gaussian surface whose center is at $x = b$.

From Problem 83:

$$\vec{\boldsymbol{E}}_\rho + \vec{\boldsymbol{E}}_{-\rho} = \frac{\rho \boldsymbol{b}}{3\in_0}\hat{\boldsymbol{i}}$$

Substituting in equation (1) yields:

$$\vec{\boldsymbol{E}} = \frac{\rho \boldsymbol{b}}{3\in_0}\hat{\boldsymbol{i}} + \vec{\boldsymbol{E}}_Q$$

Assuming that the cavity is filled with positive charge $Q$:

$$\vec{\boldsymbol{E}} = \frac{\rho \boldsymbol{b}}{3\in_0}\hat{\boldsymbol{i}} + \frac{\boldsymbol{Q}}{4\boldsymbol{\pi}\in_0 \boldsymbol{b}^3}\boldsymbol{r}'\hat{\boldsymbol{r}}$$

The vectors $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}' = r'\hat{\boldsymbol{r}}'$ are given by:

$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and $\vec{\boldsymbol{r}}' = (x-b)\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$
where $x$ and $y$ are the coordinates of any point in the cavity.

Substitute for $r\hat{\boldsymbol{r}}$ and $r'\hat{\boldsymbol{r}}'$ and simplify to obtain:

$$\vec{\boldsymbol{E}} = \frac{\rho \boldsymbol{b}}{3 \in_0}\hat{\boldsymbol{i}} + \frac{\boldsymbol{Q}}{4\pi \in_0 \boldsymbol{b}^3}\left[(\boldsymbol{x}-\boldsymbol{b})\hat{\boldsymbol{i}} + \boldsymbol{y}\hat{\boldsymbol{j}}\right]$$

At point 1, $x = 2b$ and $y = 0$:

$$\vec{\boldsymbol{E}}(2\boldsymbol{b},0) = \frac{\rho \boldsymbol{b}}{3 \in_0}\hat{\boldsymbol{i}} - \frac{\boldsymbol{Q}}{4\pi \in_0 \boldsymbol{b}^3}\left[(2\boldsymbol{b}-\boldsymbol{b})\hat{\boldsymbol{i}}\right] = \boxed{\left(\frac{\rho \boldsymbol{b}}{3 \in_0} + \frac{\boldsymbol{Q}}{4\pi \in_0 \boldsymbol{b}^2}\right)\hat{\boldsymbol{i}}}$$

At point 2, $x = 0$ and $y = 0$:

$$\vec{\boldsymbol{E}}(0,0) = \frac{\rho \boldsymbol{b}}{3 \in_0}\hat{\boldsymbol{i}} + \frac{\boldsymbol{Q}}{4\pi \in_0 \boldsymbol{b}^3}\left[(-\boldsymbol{b})\hat{\boldsymbol{i}}\right] = \boxed{\left(\frac{\rho \boldsymbol{b}}{3 \in_0} - \frac{\boldsymbol{Q}}{4\pi \in_0 \boldsymbol{b}^2}\right)\hat{\boldsymbol{i}}}$$

**86** ••• A *small* Gaussian surface in the shape of a cube has faces parallel to the *xy*, *xz*, and *yz* planes (Figure 22-47) and is in a region in which the electric field is parallel to the *x* axis. (*a*) Using the differential approximation, show that the net electric flux of the electric field out of the Gaussian surface is given by $\phi_{\text{net}} \approx \frac{\partial E_x}{\partial x}\Delta V$ , where $\Delta V$ is the volume enclosed by the Gaussian surface.

(*b*) Using Gauss's law and the results of Part (*a*) show that $\frac{\partial E_x}{\partial x} = \frac{\rho}{\epsilon_0}$, where $\rho$ is the volume charge density inside the cube. (This equation is the one-dimensional version of the point form of Gauss's law.)

**Picture the Problem** Let the coordinates of one corner of the cube be (*x,y,z*), and assume that the sides of the cube are $\Delta x$, $\Delta y$, and $\Delta z$ and compute the flux through the faces of the cube that are parallel to the *yz* plane. The net flux of the electric field out of the gaussian surface is the difference between the flux out of the surface and the flux into the surface.

(*a*) The net flux out of the cube is given by:

$$\phi_{\text{net}} = \phi(x+\Delta x) - \phi(x)$$

Use a Taylor series expansion to express the net flux through faces of the cube that are parallel to the *yz* plane:

$$\phi_{\text{net}} = \phi(x) + (\Delta x)\phi'(x) + \tfrac{1}{2}(\Delta x)^2\phi''(x) + \ldots - \phi(x) = (\Delta x)\phi'(x) + \tfrac{1}{2}(\Delta x)^2\phi''(x) + \ldots$$

Neglecting terms higher than first order we have:

$$\phi_{\text{net}} \approx \Delta x \phi'(x)$$

Because the electric field is in the $x$ direction, $\phi(x)$ is:

$$\phi(x) = E_x \Delta y \Delta z$$

and

$$\phi'(x) = \frac{\partial E_x}{\partial x}\Delta y \Delta z$$

Substitute for $\phi'(x)$ to obtain:

$$\phi_{net} \approx \Delta x \frac{\partial E_x}{\partial x}(\Delta y \Delta z) = \frac{\partial E_x}{\partial x}(\Delta x \Delta y \Delta z) = \boxed{\frac{\partial E_x}{\partial x}\Delta V}$$

(*b*) From Gauss's law, the net flux through any surface is:

$$\phi_{net} = \frac{q_{encl}}{\epsilon_0} = \frac{\rho}{\epsilon_0}\Delta V$$

From Part (*a*):

$$\phi_{net} = \frac{\partial E_x}{\partial x}\Delta V$$

Equate these two expressions and simplify to obtain:

$$\frac{\partial E_x}{\partial x}\Delta V = \frac{\rho}{\epsilon_0}\Delta V \Rightarrow \frac{\partial E_x}{\partial x} = \boxed{\frac{\rho}{\epsilon_0}}$$

**87 ••• [SSM]** Consider a simple but surprisingly accurate model for the hydrogen molecule: two positive point charges, each having charge $+e$, are placed inside a uniformly charged sphere of radius $R$, which has a charge equal to $-2e$. The two point charges are placed symmetrically, equidistant from the center of the sphere (Figure 22-48). Find the distance from the center, $a$, where the net force on either point charge is zero.

**Picture the Problem** We can find the distance from the center where the net force on either charge is zero by setting the sum of the forces acting on either point charge equal to zero. Each point charge experiences two forces; one a Coulomb force of repulsion due to the other point charge, and the second due to that fraction of the sphere's charge that is between the point charge and the center of the sphere that creates an electric field at the location of the point charge.

Apply $\sum F = 0$ to either of the point charges:

$$F_{Coulomb} - F_{field} = 0 \qquad (1)$$

Express the Coulomb force on the proton:

$$F_{Coulomb} = \frac{ke^2}{(2a)^2} = \frac{ke^2}{4a^2}$$

The force exerted by the field $E$ is:

$$F_{field} = eE$$

Apply Gauss's law to a spherical surface of radius $a$ centered at the origin:

$$E\left(4\pi a^2\right)=\frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Relate the charge density of the electron sphere to $\boldsymbol{Q}_{\text{enclosed}}$:

$$\frac{2e}{\frac{4}{3}\pi R^3}=\frac{Q_{\text{enclosed}}}{\frac{4}{3}\pi a^3}\Rightarrow Q_{\text{enclosed}}=\frac{2ea^3}{R^3}$$

Substitute for $\boldsymbol{Q}_{\text{enclosed}}$:

$$E\left(4\pi a^2\right)=\frac{2ea^3}{\epsilon_0 R^3}$$

Solve for $E$ to obtain:

$$E=\frac{ea}{2\pi\epsilon_0 R^3}\Rightarrow F_{\text{field}}=\frac{e^2a}{2\pi\epsilon_0 R^3}$$

Substitute for $\boldsymbol{F}_{\text{Coulomb}}$ and $\boldsymbol{F}_{\text{field}}$ in equation (1):

$$\frac{ke^2}{4a^2}-\frac{e^2a}{2\pi\epsilon_0 R^3}=0$$

or

$$\frac{ke^2}{4a^2}-\frac{2ke^2a}{R^3}=0\Rightarrow a=\sqrt[3]{\frac{1}{8}}R=\boxed{\tfrac{1}{2}R}$$

**88** ••• An electric dipole that has a dipole moment of $\vec{\boldsymbol{p}}$ is located at a perpendicular distance $R$ from an infinitely long line charge that has a uniform linear charge density $\lambda$. Assume that the dipole moment is in the same direction as the field of the line of charge. Determine an expression for the electric force on the dipole.

**Picture the Problem** We can find the field due to the infinitely long line charge from $E=2k\lambda/r$ and the force that acts on the dipole using $F=p\,dE/dr$.

Express the force acting on the dipole:

$$F=p\frac{dE}{dr}\qquad(1)$$

The electric field at the location of the dipole is given by:

$$E=\frac{2k\lambda}{r}$$

Substitute for $E$ in equation (1) to obtain:

$$F=p\frac{d}{dr}\left[\frac{2k\lambda}{r}\right]=\boxed{-\frac{2k\lambda p}{r^2}}$$

where the minus sign indicates that the dipole is attracted to the line charge.